

VECTOR BUNDLES ON PROJECTIVE VARIETIES WHOSE RESTRICTIONS TO AMPLE SUBVARIETIES SPLIT

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ABSTRACT. We systematically study the splitting of vector bundles on a smooth, projective variety, whose restriction to the zero locus of a regular section of an ample vector bundle splits.

First, we find ampleness and genericity conditions which ensure that the splitting of the vector bundle along the subvariety implies its global splitting. Second, we obtain a simple splitting criterion for vector bundles on the Grassmannian and on partial flag varieties.

INTRODUCTION

We say that a vector bundle splits if it is isomorphic to a direct sum of line bundles. Horrocks proved in [10] his celebrated criterion for vector bundles on projective spaces, and his ideas gave rise to two main methods for proving the splitting of a vector bundle: either by imposing cohomological conditions, or by restricting them to hypersurfaces in the ambient space.

In this article we will follow the latter path. Although there are several splitting criteria obtained by restricting to divisors, it seems there are *no similar results* for restrictions to higher co-dimensional subvarieties. Horrocks' result implies that a vector bundle on the projective space $\mathbb{P}_{\mathbf{k}}^d$ —where \mathbf{k} is an algebraically closed field and $d \geq 3$ —splits if and only if its restriction to a plane $\mathbb{P}_{\mathbf{k}}^2 \subset \mathbb{P}_{\mathbf{k}}^d$ does so. Clearly, any plane is an ample subvariety of $\mathbb{P}_{\mathbf{k}}^d$. Our goal is to generalize this observation. Given a vector bundle \mathcal{V} on a smooth, projective variety X , we ask under which assumptions the splitting of \mathcal{V} along the zero locus of a regular section of an ample vector bundle \mathcal{N} on X implies its global splitting. We investigate this issue from two points of view, each being interesting in its own right.

First we prove that, if \mathcal{N} is *sufficiently ample* compared to $\mathcal{E}nd(\mathcal{V})$, the splitting of \mathcal{V} along the zero locus Y_s of an *arbitrary* regular section $s \in \Gamma(X, \mathcal{N})$ implies its global splitting. The proof requires the vanishing of various cohomology groups, and we carefully *control* the amount of ampleness of \mathcal{N} necessary to achieve it. When the rank of \mathcal{N} is low compared to the dimension of X , our criteria take simple form, making them suited for concrete applications.

Second, we avoid imposing ampleness on \mathcal{N} ; rather we focus on the *genericity* of the section $s \in \Gamma(X, \mathcal{N})$. The splitting of \mathcal{V} along the zero locus $Y_s \subset X$ implies its global splitting under the following hypotheses:

- (i) the ample vector bundle \mathcal{N} is globally generated, its rank is less than $\frac{1}{3} \dim X$ (cf. (4.1)), and s is *very general* (in a precise sense);
- (ii) either the first cohomology group of any line bundle on X vanishes, or the (finite dimensional) algebra of endomorphisms $\mathcal{V} \otimes \mathcal{O}_{Y_s}$ is semi-simple.

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If only (i) is fulfilled, then \mathcal{V} is a successive extension of line bundles on X .

In the last section we obtain a simple splitting criterion for vector bundles on the Grassmannian $\mathrm{Gr}(e; \mathbf{k}^d)$ and on the partial flag variety $\mathrm{Fl}(d_1, \dots, d_t; \mathbf{k}^{d_{t+1}})$: a vector bundle splits if and only if, respectively, its restriction to an embedded $\mathrm{Gr}(2, \mathbf{k}^4)$ and $\mathrm{Fl}(2, \dots, 2t; \mathbf{k}^{2(t+1)})$ splits. We point out that, to our knowledge, currently there are *no splitting criteria* (cohomological, uniformity, *etc.*) for partial flag varieties.

Let us elaborate on the results. Sufficient conditions which allow to extend the splitting of a vector bundle from a subvariety to the ambient space are given in the Proposition 2.5. It is the common root of the results obtained in this article and states *without restrictions* that \mathcal{V} splits on X if and only if it does on the m^{th} -order thickening $Y_{s,m}$ of the subvariety Y_s , of dimension at least one, for $m \gg 0$. Care is taken to make this statement *effective*: we determine the order of the thickening of Y_s in X for which the splitting of \mathcal{V} along $Y_{s,m}$ implies its global splitting. The proofs rely on the Buchsbaum-Eisenbud generic free resolutions [6], combined with the vanishing theorems of Laytimi [12] and Manivel [14]. For Y smooth, we significantly improve the Proposition in two directions (cf. Theorem 2.6): first, we state it *intrinsically* for subvarieties $Y \subset X$ with ample normal bundle (thus justifying the term ‘ample subvariety’ used in the title); second, we improve the previous bound.

The Sections 3, 4 investigate the possibility of restricting \mathcal{V} to the zero locus Y_s itself, rather than to its thickening. The conclusions are contained in the Theorems 3.1 and 4.9: the former imposes *ampleness conditions* for \mathcal{N} which are sufficient to restrict to zero loci of *arbitrary* sections; the latter imposes *genericity conditions* for s , and holds for \mathcal{N} ample, globally generated.

Despite sharing a common root, the proofs of these results are *very different* in nature. The cohomological criterion is based on *effective cohomology vanishing* theorems; the genericity criterion is essentially a *gluing* argument. We illustrate our results with concrete examples.

The last section discusses the necessity of the overall assumption that the vector bundle \mathcal{N} , where we take sections, is ample. By analysing the case of the Grassmannian and of the partial flag varieties, we conclude that the hypothesis can be weakened if \mathcal{N} is globally generated and one has enough control on the cohomology of the zero locus Y_s . The main results are the Theorems 6.1 and 6.4, respectively, which massively simplify the complexity of the problem concerning the splitting of the vector bundles. The existing cohomological criteria [16, 13] for the Grassmannian involve a *large* number of tests and, as far as we know, similar results are missing for flag varieties.

1. THE FRAMEWORK AND THE APPROACH TO THE PROBLEM

Notation 1.1 Throughout the article X stands for an irreducible, smooth, projective variety, defined over an algebraically closed field \mathbf{k} of characteristic zero. For a closed subscheme $S \subset X$, we denote by $\mathcal{I}_S \subset \mathcal{O}_X$ its sheaf of ideals. A *vector (resp. line) bundle* stands for a *locally free (resp. invertible) sheaf*.

We consider two vector bundles \mathcal{V} and \mathcal{N} on X , of rank r and ν respectively, and assume that \mathcal{N} is ample. Let $\mathcal{E} := \mathcal{E}nd(\mathcal{V})$ be the bundle of endomorphisms of \mathcal{V} , $\mathcal{V}_S := \mathcal{V} \otimes \mathcal{O}_S$, and similarly $\mathcal{E}_S := \mathcal{E} \otimes \mathcal{O}_S$.

The zero locus of a section $s \in \Gamma(X, \mathcal{N})$ is the subscheme Y_s of X defined by the ideal sheaf $\mathcal{I}_{Y_s} := \mathrm{Image}(\mathcal{N}^\vee \xrightarrow{s} \mathcal{O}_X)$. We say that s is a *regular section* if its zero locus Y_s is a locally complete intersection in X .

Let $Y \subset X$ be the zero locus of a regular section of \mathcal{N} , whose ideal sheaf is \mathcal{I}_Y . For $m \geq 0$, the m -th order thickening Y_m of Y is the closed subscheme defined by \mathcal{I}_Y^{m+1} ; note that $Y_0 = Y$ with this convention. The structure sheaves of two consecutive thickenings of Y fit into the exact sequence

$$0 \rightarrow \mathrm{Sym}^m(\mathcal{N}_Y^\vee) \cong \mathcal{I}_Y^m / \mathcal{I}_Y^{m+1} \rightarrow \mathcal{O}_{Y_m} \rightarrow \mathcal{O}_{Y_{m-1}} \rightarrow 0. \quad (1.1)$$

Definition 1.2 The *formal completion* of X along Y is defined as the direct limit $\varinjlim Y_m$, and it is denoted \hat{X}_Y . If no confusion is possible, we write \hat{X} .

When the ground field $\mathbf{k} = \mathbb{C}$, there are two kinds of thickenings and completions: one using germs of regular functions and another using germs of analytic functions. However, the generators of \mathcal{I}_Y^m are the same for all m , by Chow's theorem, so the exact sequence (1.1) is valid in both cases.

Definition 1.3 We say that the vector bundle \mathcal{V} *splits* (or *is split*) if there are r line sub-bundles $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ of \mathcal{V} such that $\mathcal{V} = \bigoplus_{i=1}^r \mathcal{L}_i$. Thus \mathcal{V} splits if and only if there are pairwise non-isomorphic line sub-bundles \mathcal{L}_j , $j \in J$, of \mathcal{V} such that

$$\mathcal{V} = \bigoplus_{j \in J} \mathcal{L}_j \otimes \mathbf{k}^{m_j} \quad \text{with} \quad \sum_{j \in J} m_j = r. \quad (1.2)$$

We call the vector sub-bundles $\mathcal{V}_j := \mathcal{L}_j \otimes \mathbf{k}^{m_j}$, $j \in J$, the *isotypical components* of \mathcal{V} corresponding to the splitting (1.2).

If $\bigoplus_{j \in J} \mathcal{L}_j \otimes \mathbf{k}^{m_j}$ and $\bigoplus_{j' \in J'} \mathcal{L}'_{j'} \otimes \mathbf{k}^{m'_{j'}}$ are two splittings of \mathcal{V} , then there is a bijective function $\sigma : J \rightarrow J'$ such that $\mathcal{L}'_{\sigma(j)} \cong \mathcal{L}_j$ and $m'_{\sigma(j)} = m_j$ for all $j \in J$. (See [2, Theorem 1 and 2].)

Unfortunately, the isotypical components are *not uniquely defined*, they depend on the choice of the splitting. Indeed, the global automorphisms of \mathcal{V} send a splitting into another one. We define the following relation on the index set J :

$$i \prec j \quad \Leftrightarrow \quad i \neq j \text{ and } \Gamma(X, \mathcal{L}_i^{-1} \mathcal{L}_j) \neq 0. \quad (1.3)$$

It is straightforward to check that ' \prec ' is a partial order. The maximal elements with respect to \prec have the property that the corresponding isotypical components are uniquely defined.

Lemma 1.4 *Let $M \subset J$ be the subset of maximal elements with respect to \prec . Then there is a natural, injective homomorphism of vector bundles*

$$\mathrm{ev}_M : \bigoplus_{j \in M} \mathcal{L}_j \otimes \Gamma(X, \mathcal{L}_j^{-1} \mathcal{V}) \rightarrow \mathcal{V}. \quad (1.4)$$

Proof. Clear, by the very definition. □

In this article we attempt to address the following:

Question. Let Y be the zero set of a regular section s of \mathcal{N} and assume that \mathcal{V}_Y splits. When does \mathcal{V} split too?

The key to test the splitting of a vector bundle is the following elementary lemma, which allows to lift the splitting along a subvariety to the ambient space.

Lemma 1.6 *Let $S \subset X$ be a closed subscheme such that \mathcal{V}_S splits and*

$$\mathrm{res}_S : \Gamma(X, \mathcal{E}) \rightarrow \Gamma(S, \mathcal{E}_S)$$

is surjective—in particular, $H^1(\mathcal{I}_S \otimes \mathcal{E}) = 0$. Then \mathcal{V} splits.

Proof. The hypothesis says that $\mathcal{V}_S \cong \ell_1 \oplus \dots \oplus \ell_r$, where $r := \text{rk}(\mathcal{V})$ and $\ell_1, \dots, \ell_r \in \text{Pic}(S)$. Take $\varepsilon_1, \dots, \varepsilon_r \in \mathbf{k}$ pairwise distinct, and consider $\phi \in \text{End}(V_S)$ given by multiplication by ε_ρ on ℓ_ρ . Since res_S is surjective, ϕ extends to $\Phi \in \Gamma(X, \mathcal{E})$. But the eigenvalues of Φ are independent of $x \in X$, so they are precisely $\varepsilon_1, \dots, \varepsilon_r$. Overall, we obtain $\Phi \in \text{End}(\mathcal{V})$ with $\text{rk}(\mathcal{V})$ distinct eigenvalues. Hence Φ_x is diagonalizable for all $x \in X$, and $\mathcal{L}_\rho := \text{Ker}(\varepsilon_\rho \mathbb{1} - \Phi)$ are line bundles on X such that $\mathcal{V} = \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_r$. \square

Strategy To answer the question above we should show the surjectivity of the homomorphism $\text{res}_Y : \Gamma(X, \mathcal{E}) \rightarrow \Gamma(Y, \mathcal{E}_Y)$. This is done in two stages:

- (i) Prove that $\Gamma(X, \mathcal{E}) \rightarrow \Gamma(Y_m, \mathcal{E}_{Y_m})$ is surjective, for $m \gg 0$. Actually, we will determine *effective lower bounds* for m such that $H^1(\mathcal{E} \otimes \mathcal{I}_Y^m) = 0$.
- (ii) Prove that $\Gamma(Y_m, \mathcal{E}_{Y_m}) \rightarrow \Gamma(Y, \mathcal{E}_Y)$ is surjective, for all $m \geq 1$.

In this article, we will repeatedly use base change arguments: the property of a vector bundle to be split is unaffected by changing the (algebraically closed) ground field. The proposition below can be interpreted as the invariance of the Krull-Schmidt decomposition (see [2]) under the change of the ground field.

Proposition 1.7 *Let \mathbf{h}, \mathbf{k} be two algebraically closed fields of characteristic zero, such that $\mathbf{h} \subset \mathbf{k}$. Consider an irreducible projective scheme $X_{\mathbf{h}}$ over \mathbf{h} , and $\mathcal{V}_{\mathbf{h}}$ a vector bundle on it. We define $X_{\mathbf{k}} := X_{\mathbf{h}} \times_{\mathbf{h}} \mathbf{k}$ and $\mathcal{V}_{\mathbf{k}} := \mathcal{V}_{\mathbf{h}} \times_{X_{\mathbf{h}}} X_{\mathbf{k}}$. Then $\mathcal{V}_{\mathbf{h}}$ splits if and only if $\mathcal{V}_{\mathbf{k}}$ splits.*

Proof. We denote by $\text{Aut}(\mathcal{V}_{\mathbf{k}}) \subset \text{End}(\mathcal{V}_{\mathbf{k}})$ the automorphism group of $\mathcal{V}_{\mathbf{k}}$, and similarly for $\mathcal{V}_{\mathbf{h}}$. Then $\text{Aut}(\mathcal{V}_{\mathbf{k}})$ and $\text{Aut}(\mathcal{V}_{\mathbf{h}})$ are linear algebraic groups and $\text{Aut}(\mathcal{V}_{\mathbf{k}}) = \text{Aut}(\mathcal{V}_{\mathbf{h}}) \otimes \mathbf{k}$, by base change. Obviously, $\mathcal{V}_{\mathbf{k}}$ splits if and only if the dimension of the maximal torus of $\text{Aut}(\mathcal{V}_{\mathbf{k}})$ equals $\text{rk}(\mathcal{V}_{\mathbf{k}})$, and the same holds for $\mathcal{V}_{\mathbf{h}}$. But the dimension of the maximal torus is preserved under base change, and the conclusion follows. \square

2. THE SURJECTIVITY OF $\Gamma(X, \mathcal{E}) \rightarrow \Gamma(Y_m, \mathcal{E}_{Y_m})$ AND EFFECTIVE LOWER BOUNDS FOR m

Let $Y := Y_s \subset X$ be the zero locus of a regular section $s \in \Gamma(X, \mathcal{N})$. Then its sheaf of ideals $\mathcal{I}_Y \subset \mathcal{O}_X$ admits the following well-known Koszul resolution:

$$0 \longrightarrow \bigwedge^{\nu} \mathcal{N}^{\vee} \xrightarrow{s \lrcorner} \bigwedge^{\nu-1} \mathcal{N}^{\vee} \longrightarrow \dots \longrightarrow \mathcal{N}^{\vee} \xrightarrow{s \lrcorner} \mathcal{I}_Y \longrightarrow 0. \quad (2.1)$$

(Here \lrcorner stands for the contraction operation.) More generally, locally free resolutions of the powers of \mathcal{I}_Y are constructed in [6, Theorem 3.1]. For any $m \geq 1$, we have the resolution

$$0 \rightarrow L_m^{\nu}(\mathcal{N}^{\vee}) \rightarrow L_m^{\nu-1}(\mathcal{N}^{\vee}) \rightarrow \dots \rightarrow L_m^j(\mathcal{N}^{\vee}) \rightarrow \dots \rightarrow \text{Sym}^m(\mathcal{N}^{\vee}) \xrightarrow{s^m \lrcorner} \mathcal{I}_Y^m \rightarrow 0, \quad (2.2)$$

where the vector bundles $L_m^j(\mathcal{N}^{\vee})$, $1 \leq j \leq \nu$, are defined as follows:

$$\begin{aligned} L_m^j(\mathcal{N}^{\vee}) &:= \text{Ker} \left(\text{Sym}^m(\mathcal{N}^{\vee}) \otimes \bigwedge^{j-1} \mathcal{N}^{\vee} \longrightarrow \text{Sym}^{m+1}(\mathcal{N}^{\vee}) \otimes \bigwedge^{j-2} \mathcal{N}^{\vee} \right) \\ &= \text{Im} \left(\text{Sym}^{m-1}(\mathcal{N}^{\vee}) \otimes \bigwedge^j \mathcal{N}^{\vee} \longrightarrow \text{Sym}^m(\mathcal{N}^{\vee}) \otimes \bigwedge^{j-1} \mathcal{N}^{\vee} \right). \end{aligned} \quad (2.3)$$

Actually $L_m^j(\mathcal{N}^{\vee})$ is a direct summand in both $\text{Sym}^m(\mathcal{N}^{\vee}) \otimes \bigwedge^{j-1} \mathcal{N}^{\vee}$ and $\text{Sym}^{m-1}(\mathcal{N}^{\vee}) \otimes \bigwedge^j \mathcal{N}^{\vee}$ because the homomorphisms which define $L_m^j(\mathcal{N}^{\vee})$ are $\mathcal{A}ut(\mathcal{N}^{\vee})$ -invariant and the general

linear group is linearly reductive. The long exact sequence (2.2) breaks up into $\nu - 1$ short exact sequences of the form

$$0 \rightarrow \mathcal{S}_{j+1}^{(m)} \rightarrow L_m^j(\mathcal{N}^\vee) \rightarrow \mathcal{S}_j^{(m)} \rightarrow 0, \quad j = 1, \dots, \nu - 1, \quad (2.4)$$

with $\mathcal{S}_1^{(m)} = \mathcal{I}_Y^m$ and $\mathcal{S}_\nu^{(m)} = \text{Sym}^{m-1}(\mathcal{N}^\vee) \otimes \det(\mathcal{N}^\vee)$.

Lemma 2.1 *Let \mathcal{F} be a vector bundle of rank f on X .*

(i) (arbitrary ν , lot of positivity for \mathcal{N})

Let $m \geq 0$ be such that $\text{Sym}^{1+f}(\mathcal{F}^\vee) \otimes \det(\mathcal{F}) \otimes \text{Sym}^{m+\nu}(\mathcal{N}) \otimes \det(\mathcal{N})^{-1}$ is ample. Then holds

$$H^t(X, \mathcal{F} \otimes \text{Sym}^m(\mathcal{N}^\vee) \otimes \bigwedge^j \mathcal{N}^\vee) = 0, \quad \forall t < \dim X - \nu + j.$$

In particular, if $\text{Sym}^{1+f}(\mathcal{F}^\vee) \otimes \det(\mathcal{F}) \otimes \text{Sym}^{1+\nu}(\mathcal{N}) \otimes \det(\mathcal{N})^{-1}$ is ample and $\nu \leq \dim X - 2$,

then $H^j(X, \mathcal{F} \otimes \bigwedge^j \mathcal{N}^\vee) = 0, \forall j = 1, \dots, \nu$.

(ii) (low ν , little positivity for \mathcal{N})

If $\mathcal{F}^\vee \otimes \mathcal{N}$ is ample and $\frac{(\nu+1)^2}{4} \leq \dim X - f$, then $H^j(\mathcal{F} \otimes \bigwedge^j \mathcal{N}^\vee) = 0$, for all $j = 1, \dots, \nu$.

Proof. (i) We consider the diagram

$$\begin{array}{ccc} & Z & \\ \swarrow & \downarrow p & \searrow \\ \mathbb{P}(\mathcal{F}) & & \mathbb{P}(\mathcal{N}^\vee) \\ \searrow p_F & & \swarrow p_N \\ & X & \end{array}$$

where $Z := \mathbb{P}(\mathcal{F}) \times_X \mathbb{P}(\mathcal{N}^\vee)$ and $\mathcal{O}_{p_F}(1)$, $\mathcal{O}_{p_N}(1)$ stand for the relatively ample line bundles on $\mathbb{P}(\mathcal{F})$, $\mathbb{P}(\mathcal{N}^\vee)$, respectively. Then the relative canonical bundle of p_N satisfies

$$\kappa_{p_N} = \det(\mathcal{N}) \otimes \mathcal{O}_{p_N}(-\nu), \text{ so}$$

$$\text{Sym}^m(\mathcal{N}) = (p_N)_*(\kappa_{p_N} \otimes \mathcal{O}_{p_N}(m + \nu) \otimes \det(\mathcal{N})^{-1}), \quad \forall m \geq 0.$$

Similar conclusion holds for the relative canonical bundle κ_{p_F} of p_F .

By hypothesis, $\mathcal{L} := (\mathcal{O}_{p_F}(1 + f) \otimes \det(\mathcal{F})) \boxtimes (\mathcal{O}_{p_N}(m + \nu) \otimes \det(\mathcal{N})^{-1})$ on Z is ample, and $\kappa_X \otimes \mathcal{F}^\vee \otimes \text{Sym}^m(\mathcal{N}) = p_*(\kappa_Z \otimes \mathcal{L})$; the projection formula implies

$$H^i(X, \kappa_X \otimes \mathcal{F}^\vee \otimes \text{Sym}^m(\mathcal{N}) \otimes \bigwedge^j \mathcal{N}) = H^i(Z, \kappa_Z \otimes \mathcal{L} \otimes \bigwedge^j p^* \mathcal{N}).$$

On the right-hand-side, $p^* \mathcal{N}$ is nef and \mathcal{L} is ample, so [14] implies that the cohomology group above vanishes for $i > \nu - j$. We conclude by applying the Serre duality on X .

For the second claim: $H^j(\mathcal{F}^\vee \otimes \mathcal{N}^\vee \otimes \bigwedge^{j-1} \mathcal{N}^\vee) = 0$, since $j < \dim X - \nu + j - 1$ and $\bigwedge^j \mathcal{N}$ is a direct summand in $\mathcal{N} \otimes \bigwedge^{j-1} \mathcal{N}$.

(ii) If $\mathcal{F}^\vee \otimes \mathcal{N}$ is ample, then $\mathcal{F}^\vee \otimes \bigwedge^j \mathcal{N}$ is ample too, being a direct summand of $\mathcal{F}^\vee \otimes \mathcal{N}^{\otimes j}$.

Hence [12, Theorem 2.1] yields $H^{\dim X - j}(X, \kappa_X \otimes \mathcal{F}^\vee \otimes \bigwedge^j \mathcal{N}) = 0$, for all $j = 1, \dots, \nu$. \square

Proposition 2.2 *Let \mathcal{N} be an ample vector bundle of rank ν on X and let $s \in \Gamma(X, \mathcal{N})$ be a regular section with zero locus Y . We consider an arbitrary, locally free sheaf \mathcal{F} of rank f on X . Then the following hold:*

(i) Let $m_{\mathcal{F}} \geq 1$ be minimal such that

$$\mathrm{Sym}^{1+f}(\mathcal{F}^\vee) \otimes \det(\mathcal{F}) \otimes \mathrm{Sym}^{m_{\mathcal{F}}-1+\nu}(\mathcal{N}) \otimes \det(\mathcal{N})^{-1}$$

is ample. Then we have: $H^t(X, \mathcal{F} \otimes \mathcal{I}_Y^m) = 0, \forall m \geq m_{\mathcal{F}}, \forall t \leq \dim Y = \dim X - \nu$.

In particular, $H^j(X, \mathcal{F}) \rightarrow H^j(Y_m, \mathcal{F})$ is an isomorphism, for $0 \leq j \leq \dim Y - 1$, $m \gg 0$.

(ii) Assume $f + \frac{(\nu+1)^2}{4} \leq \dim X$, and $\mathcal{F}^\vee \otimes \mathcal{N}$ is ample. Then $H^1(X, \mathcal{F} \otimes \mathcal{I}_Y) = 0$.

Proof. (i) We tensor (2.4) by \mathcal{F} . Since the middle term $\mathcal{F} \otimes L_m^j(\mathcal{N}^\vee)$ is a direct summand in $\mathcal{F} \otimes \mathrm{Sym}^{m-1}(\mathcal{N}^\vee) \otimes \bigwedge^j \mathcal{N}^\vee$ for all j , the Lemma 2.1(i) implies that for all $t \leq \dim X - \nu$ holds $H^{t+j-1}(\mathcal{F} \otimes L_m^j(\mathcal{N}^\vee)) = 0$, hence

$$H^t(\mathcal{F} \otimes \mathcal{I}_Y^m) \subset H^{t+1}(\mathcal{F} \otimes \mathcal{S}_2^{(m)}) \subset \dots \subset H^{t+\nu-2}(\mathcal{F} \otimes \mathcal{S}_{\nu-1}^{(m)}) \subset H^{t+\nu-1}(\mathcal{F} \otimes L_m^\nu(\mathcal{N}^\vee)) = 0.$$

(ii) The same argument as above, together with the Lemma 2.1(ii) yields:

$$H^1(X, \mathcal{F} \otimes \mathcal{I}_Y) \subset H^2(X, \mathcal{F} \otimes \mathcal{S}_2^{(1)}) \subset \dots \subset H^\nu(X, \mathcal{F} \otimes \det \mathcal{N}^\vee) = 0. \quad \square$$

Corollary 2.3 Let $\mathbf{k} = \mathbb{C}$, and the situation be as in 2.2, with $\dim Y \geq 2$. Consider an (analytic or Zariski) open neighbourhood \mathcal{U} of Y in X , and let \mathcal{A}, \mathcal{C} be two vector bundles on X . Then the following statements hold:

(i) Any extension of vector bundles on \mathcal{U} ,

$$0 \rightarrow \mathcal{A} \otimes \mathcal{O}_{\mathcal{U}} \rightarrow \mathcal{G}_{\mathcal{U}} \rightarrow \mathcal{C} \otimes \mathcal{O}_{\mathcal{U}} \rightarrow 0, \quad (\mathrm{G})$$

can be extended to an extension $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$ of vector bundles on X , and \mathcal{B} is uniquely defined, up to isomorphism.

(ii) Assume that the restriction to \mathcal{U} of three vector bundles $\mathcal{A}, \mathcal{C}, \mathcal{G}$ on X fit in the extension (G). Then \mathcal{G} is an extension of \mathcal{C} by \mathcal{A} on X .

Proof. (i) Regardless whether the computations are done algebraically or analytically, the resolution (2.2) and the Proposition 2.2 are both valid.

The extension (G) corresponds to $\eta_{\mathcal{U}} \in H^1(\mathcal{U}, \mathcal{C}^\vee \otimes \mathcal{A})$, and its restriction to Y_m corresponds to the image $\eta_m \in H^1(Y_m, \mathcal{C}^\vee \otimes \mathcal{A})$ of $\eta_{\mathcal{U}}$. The Proposition 2.2(i) implies that $H^1(X, \mathcal{C}^\vee \otimes \mathcal{A}) \rightarrow H^1(Y_m, \mathcal{C}^\vee \otimes \mathcal{A})$, $m \gg 0$, is an isomorphism, so η_m uniquely lifts to $\eta \in H^1(X, \mathcal{C}^\vee \otimes \mathcal{A})$; this defines the extension $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$ on X . For the uniqueness part, notice that any two vector bundles \mathcal{B} and \mathcal{B}' are isomorphic along Y_m and apply 2.2 again.

(ii) It is just a reformulation of the uniqueness statement above. \square

Lemma 2.4 Assume that $\mathcal{V}_{Y_{m_0}}$ splits, for some $m_0 \geq 0$. Then \mathcal{V} splits as soon as either one of the following conditions is satisfied:

(i) $\nu \leq \dim X - 1$, and $H^j(X, \mathcal{E} \otimes \mathrm{Sym}^{m_0}(\mathcal{N}^\vee) \otimes \bigwedge^j \mathcal{N}^\vee) = 0$, for $j = 1, \dots, \nu$.
or (ii) $\nu \leq \dim X - 2$, and $H^1(Y, \mathrm{Sym}^m(\mathcal{N}_Y^\vee) \otimes \mathcal{E}_Y) = 0$, for all $m \geq m_0 + 1$.

The twist of any vector bundle by a sufficiently ample line bundle satisfies the previous conditions. Horrocks' splitting criterion for $\mathbb{P}^{\nu+2}$ is a particular case: just take $\mathcal{N} := \mathcal{O}_{\mathbb{P}^d}(1)^{\oplus \nu}$. The condition (ii) involves only \mathcal{E}_Y , which is a direct sum of line bundles; this simplifies the computations.

Proof. (i) The hypothesis implies that $H^1(X, \mathcal{E} \otimes \mathcal{I}_Y^{m_0+1}) = 0$. (See the proof of 2.2(i) above.)
(ii) The exact sequence (1.1) implies that $\text{res}_{Y_{m-1}}^{Y_m}: \Gamma(Y_m, \mathcal{E}) \rightarrow \Gamma(Y_{m-1}, \mathcal{E})$ is surjective, for all $m \geq m_0 + 1$. Hence $\Gamma(X, \mathcal{E}) \rightarrow \Gamma(Y, \mathcal{E}_{Y_{m_0}})$ is surjective too, by 2.2(i). \square

The following general statement holds for arbitrary vector bundles on projective varieties. It is the common root of the subsequent results in this article.

Proposition 2.5 *Let \mathcal{N} be an ample vector bundle of rank $\nu \leq \dim X - 1$, and let Y be the zero locus of a regular section of \mathcal{N} .*

(i) *Let $m_{\mathcal{V}} \geq 0$ be minimal such that $\text{Sym}^{1+r^2}(\mathcal{E}) \otimes \text{Sym}^{m_{\mathcal{V}}+\nu}(\mathcal{N}) \otimes \det(\mathcal{N})^{-1}$ is ample. Then \mathcal{V} splits if and only if \mathcal{V}_{Y_m} does, for some $m \geq m_{\mathcal{V}}$.*

In particular, \mathcal{V} is split if and only if its restriction to \hat{X}_Y splits.

(ii) *Assume that the ground field is $\mathbf{k} = \mathbb{C}$. Then \mathcal{V} splits if and only if there is an open analytic neighbourhood \mathcal{U} of Y such that $\mathcal{V} \otimes \mathcal{O}_{\mathcal{U}}$ splits.*

Proof. Apply the Proposition 2.2 and the Lemma 1.6. \square

When the subvariety $Y \subset X$ is smooth, the bounds in previous statement can be improved, by using the bootstrapping argument 2.4(ii) combined with the Kodaira vanishing theorem. Notably, the statement below is *intrinsic* to Y , *does not involve* the vector bundle \mathcal{N} , thus it justifies the term of ‘ample subvariety’ used in the title.

Theorem 2.6 *Let $Y \subset X$ be a smooth, irreducible subvariety and denote by \mathcal{N}_Y its normal bundle. We assume that the following hypotheses are satisfied:*

- (i) (a) $\nu := \text{codim}_X Y \leq \dim X - 2$;
(b) \mathcal{N}_Y and $\mathcal{E}_Y \otimes \text{Sym}^{m_0+1+\nu}(\mathcal{N}_Y) \otimes (\det \mathcal{N}_Y)^{-1}$ are ample.
- (ii) *The cohomological dimension of the complement is $\text{cd}(X \setminus Y) \leq \dim X - 2$.
The inequality holds if Y is the zero locus of a regular section in an ample vector bundle \mathcal{N} on X , of rank $\nu \leq \dim X - 1$.*

Then \mathcal{V} splits if and only if $\mathcal{V}_{Y_{m_0}}$ does.

Proof. We prove that (a) and (b) imply $H^1(Y, \text{Sym}^m(\mathcal{N}_Y^\vee) \otimes \mathcal{E}_Y) = 0$, for all $m \geq m_0 + 1$, so $\Gamma(\hat{X}_Y, \mathcal{E}_{\hat{X}_Y}) = \varprojlim_m \Gamma(Y_m, \mathcal{E}) \rightarrow \Gamma(Y, \mathcal{E}_Y)$ is surjective (cf. (1.1)). The projection formula yields:

$$H^1(Y, \text{Sym}^m(\mathcal{N}_Y^\vee) \otimes \mathcal{E}_Y) \cong H^\nu(\mathbb{P}(\mathcal{N}_Y^\vee), (\mathcal{O}_q(m+\nu) \otimes (q^* \det \mathcal{N}_Y)^{-1} \otimes q^* \mathcal{E}_Y)^\vee).$$

For $\mathcal{V}_Y \cong \bigoplus_{j=1}^r \ell_j$, we have $\mathcal{E}_Y \cong \bigoplus_{i,j} \ell_j \ell_i^{-1}$, hence

$$\begin{aligned} & H^\nu(\mathbb{P}(\mathcal{N}_Y^\vee), (\mathcal{O}_q(m+\nu) \otimes (q^* \det \mathcal{N}_Y)^{-1} \otimes q^* \mathcal{E}_Y)^\vee) \\ &= \bigoplus_{\ell} H^\nu(\mathbb{P}(\mathcal{N}_Y^\vee), (\mathcal{O}_q(m+\nu) \otimes (q^* \det \mathcal{N}_Y)^{-1} \otimes q^* \ell)^{-1}). \end{aligned}$$

where ℓ runs over the direct summands of \mathcal{E}_Y . The terms on the right hand side vanish, because $\mathcal{O}_q(m+\nu) \otimes (q^* \det \mathcal{N}_Y)^{-1} \otimes q^* \ell$ is ample for $m \geq m_0 + 1$.

The assumption (ii) implies that $\Gamma(X, \mathcal{E}) \rightarrow \Gamma(\hat{X}_Y, \mathcal{E}_{\hat{X}_Y})$ is surjective, by [8, Theorem III.3.4(b)], so overall $\Gamma(X, \mathcal{E}) \rightarrow \Gamma(Y_{m_0}, \mathcal{E})$ is surjective too.

Finally, the claim about the condition (ii) of the theorem follows from [8, Theorem III.3.4(b)] and the Proposition 2.2(i). \square

In the subsequent sections we will obtain sufficient conditions for these criteria.

3. THE SURJECTIVITY OF $\Gamma(Y_m, \mathcal{E}_{Y_m}) \rightarrow \Gamma(Y, \mathcal{E}_Y)$ AND 1ST CRITERION: AMPLENESS CONDITIONS FOR \mathcal{N}

In this section we consider *arbitrary* regular sections of \mathcal{N} such that \mathcal{V} splits along their zero locus, and we impose *sufficient ampleness* on \mathcal{N} in order to deduce the global splitting of \mathcal{V} .

Theorem 3.1 *The implication $[\mathcal{V}_Y \text{ splits} \Rightarrow \mathcal{V} \text{ splits}]$ holds in any of the situations described below.*

- (a) Assume that $\nu = \text{rk}(\mathcal{N}) \leq \dim X - 2$, and let $s \in \Gamma(X, \mathcal{N})$ be an arbitrary regular section with zero locus Y .
 - (i) $\text{Sym}^{1+r^2}(\mathcal{E}) \otimes \text{Sym}^{1+\nu}(\mathcal{N}) \otimes \det(\mathcal{N})^{-1}$ is ample;
 - (ii) $\frac{(\nu+1)^2}{4} \leq \dim X - r^2$ and $\mathcal{E} \otimes \mathcal{N}$ is ample;
 - (iii) Y is smooth, $\nu \leq \frac{\dim X - 1}{2}$, and $\mathcal{N} = \mathcal{G} \otimes \mathcal{A}$ with \mathcal{G} a globally generated vector bundle of rank ν , and \mathcal{A} an ample line bundle such that $\mathcal{E}_Y \otimes \mathcal{A}_Y$ is ample.
- (b) $Y \subset X$ is a smooth subvariety with the following properties:
 - (i) $\nu := \text{codim}_X Y \leq \dim X - 2$ and $\text{cd}(X \setminus Y) \leq \dim X - 2$;
 - (ii) \mathcal{N}_Y and $\mathcal{E}_Y \otimes \text{Sym}^{1+\nu}(\mathcal{N}_Y) \otimes (\det \mathcal{N}_Y)^{-1}$ are ample.

Proof. (a)(i) The claim follows from the Lemma 2.1(i) and 2.4(i).

(ii) The Propositions 2.2(ii) implies that $H^1(X, \mathcal{E} \otimes \mathcal{I}_Y) = 0$.

(iii) For all the direct summands ℓ of \mathcal{E}_Y , the line bundle $\ell \otimes \mathcal{A}_Y$ is ample. Then

$$H^1(Y, \text{Sym}^m(\mathcal{N}_Y^\vee) \otimes \mathcal{E}_Y)^\vee = \bigoplus_{\ell} H^{\dim Y - 1}(Y, \kappa_Y \otimes \underbrace{\text{Sym}^m(\mathcal{G}_Y)}_{\text{globally generated}} \otimes \underbrace{(\ell \otimes \mathcal{A}_Y^m)}_{\text{ample}}),$$

vanishes, for all $m \geq 1$, by [12, Theorem 2.4]. The Lemma 2.4(ii) yields the conclusion.

(b) This is the Theorem 2.6, for $m_0 = 0$. \square

In some cases one wishes to prove the triviality of a vector bundle (cf. [4]).

Corollary 3.2 *Assume that $\nu \leq \frac{\dim X - 1}{2}$ and $\mathcal{N} = \mathcal{G} \otimes \mathcal{A}$, with \mathcal{G} globally generated and \mathcal{A} ample. If \mathcal{V} is trivializable along the zero locus of a regular section in \mathcal{N} , and this zero locus is smooth, then \mathcal{V} is trivializable on X .*

Proof. Indeed, in this case $\mathcal{E}_Y \cong \mathcal{O}_Y^{\oplus r^2}$. \square

With the Theorem 3.1(iii) one can create a host of examples. To check the splitting of a vector bundle \mathcal{V} , one should proceed as follows: find (according to the case) a low rank, globally generated vector bundle \mathcal{G} on X , and an ample line bundle \mathcal{A} such that $\mathcal{E} \otimes \mathcal{A}$ is ample; then restrict \mathcal{V} to the zero locus of a section in $\mathcal{G} \otimes \mathcal{A}$.

Example 3.3 (Compare with 4.10(i)) Let $X \xrightarrow{\iota} \text{Gr}(n; \mathbb{C}^{n+\nu})$ be a c -codimensional subvariety, with $n \geq 3, \nu \geq 2, c \leq (n-2)\nu - 1$; let \mathcal{G} be the universal quotient bundle and $\mathcal{O}_X(1) := \det(\mathcal{G})_X$. The following statements hold:

- (i) If $\mathcal{E}(a)$ is ample for some $a \geq 1$, and \mathcal{V} splits along the (smooth) zero locus of an arbitrary regular section in $\mathcal{N} := \mathcal{G}_X(a)$, then \mathcal{V} splits on X .
- (ii) If the restriction of \mathcal{V} to the (smooth) zero locus of a regular section in $\mathcal{G}_X(1)$ is trivializable, then $\mathcal{V} \cong \mathcal{O}_X^{\oplus r}$.

Similar statements hold for the other (isotropic) Grassmannians, too.

4. A GLUING PROCEDURE AND THE 2ND CRITERION: SPLITTING ALONG ZERO LOCI OF GENERIC SECTIONS OF \mathcal{N}

Now we change our viewpoint. Instead of imposing ampleness on \mathcal{N} , we prove that the splitting of a vector bundle along the zero locus of a *very general* section of a globally generated ample vector bundle implies its global splitting. Throughout this section we assume that \mathcal{N} is *globally generated*, and furthermore:

$$\nu \leq \min\left\{\frac{\dim X - 3}{2}, \frac{\dim X - 1}{3}\right\} \text{ that is } \begin{cases} \nu = 1 & \text{for } \dim X = 5, 6, \\ \nu \leq \frac{\dim X - 1}{3} & \text{for } \dim X \geq 7, \end{cases} \quad (4.1)$$

or $\nu = 1$, $\dim X = 4$, and $\kappa_X \otimes \mathcal{N}^2$ is globally generated,
where κ_X stands for the canonical bundle of X .

Our goal is to prove that the splitting of \mathcal{V} along the geometric generic section of \mathcal{N} implies its global splitting. The proof uses base change arguments, so we start with general considerations. The variety X and the vector bundles \mathcal{N}, \mathcal{V} are defined by equations involving finitely many coefficients in \mathbf{k} . After adjoining them to \mathbb{Q} , we obtain a field extension of finite type $\mathbb{Q} \hookrightarrow \mathbf{k}_0$. In particular, \mathbf{k}_0 is countable, so we can realize it as a sub-field of \mathbb{C} .

$$\mathbf{k}_0 \hookrightarrow \mathbf{k} \xrightarrow{\mathbf{k} \text{ alg. closed}} \bar{\mathbf{k}}_0 \hookrightarrow \mathbf{k} \quad \text{and} \quad \mathbf{k}_0 \hookrightarrow \mathbb{C} \xrightarrow{\mathbb{C} \text{ alg. closed}} \bar{\mathbf{k}}_0 \hookrightarrow \mathbb{C}.$$

After replacing \mathbf{k}_0 by $\bar{\mathbf{k}}_0$, we find a countable, algebraically closed field \mathbf{k}_0 , which is simultaneously a sub-field of \mathbf{k} and of \mathbb{C} , such that $X, \mathcal{N}, \mathcal{V}$ are defined over \mathbf{k}_0 . In this situation we have the Cartesian, base change diagram

$$\begin{array}{ccc} X = X_{\mathbf{k}} & \xrightarrow{b} & X_0 := X_{\mathbf{k}_0} \\ \downarrow & & \downarrow \\ \text{Spec}(\mathbf{k}) & \longrightarrow & \text{Spec}(\mathbf{k}_0) \end{array}$$

and there are vector bundles $\mathcal{N}_0, \mathcal{V}_0$ on X_0 such that $\mathcal{N} = \mathcal{N}_0 \times_{\mathbf{k}_0} \mathbf{k}$ and also $\mathcal{V} = \mathcal{V}_0 \times_{\mathbf{k}_0} \mathbf{k}$; we denote $\mathcal{E}_0 := \mathcal{E}nd(\mathcal{V}_0)$. By base change, \mathcal{N}_0 on X_0 is globally generated too. Let $\mathbb{P}_{\mathbf{k}}^N := \mathbb{P}(\Gamma(X, \mathcal{N})) = \text{Proj}(\text{Sym}_{\mathbf{k}}^{\bullet}(\Gamma(X, \mathcal{N})^{\vee}))$, and similarly for \mathbf{k}_0 , and we consider the trace morphism

$$\mathbb{P}_{\mathbf{k}}^N \longrightarrow \mathbb{P}_{\mathbf{k}_0}^N, \quad \mathfrak{p} \longmapsto \mathfrak{p} \cap \text{Sym}_{\mathbf{k}_0}^{\bullet}(\Gamma(X_0, \mathcal{N}_0)^{\vee}).$$

The sheaf \mathcal{K} defined by

$$0 \rightarrow \mathcal{K} := \text{Ker}(\eta) \rightarrow \Gamma(X, \mathcal{N}) \otimes_{\mathcal{O}_X} \mathcal{O}_X \xrightarrow{\eta} \mathcal{N} \rightarrow 0 \quad (4.2)$$

is locally free, and the incidence variety $\mathcal{Y} := \{([s], x) \mid s(x) = 0\} \subset \mathbb{P}_{\mathbf{k}}^N \times X$ is naturally isomorphic to the projective bundle $\mathbb{P}(\mathcal{K})$ over X . We denote by π and q respectively the projections of \mathcal{Y} onto $\mathbb{P}_{\mathbf{k}}^N$ and X . For any open subset S of $\mathbb{P}_{\mathbf{k}}^N$, we let $\mathcal{Y}_S := \pi^{-1}(S)$. If the ground field $\mathbf{k} = \mathbb{C}$, we will consider open subsets of $\mathbb{P}_{\mathbb{C}}^N$ in the *analytic* topology. Henceforth we use this notation.

Definition 4.1 Let \mathbb{k} be the quotient field of $\mathbb{P}_{\mathbf{k}}^N$, and $\bar{\mathbb{k}}$ its algebraic closure. The *geometric generic section* \mathbb{Y} of \mathcal{N} is defined by the Cartesian diagram:

$$\begin{array}{ccc} \mathbb{Y} := \mathcal{Y}_{\bar{\mathbb{k}}} & \xrightarrow{\psi} & \mathcal{Y} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\bar{\mathbb{k}}) & \longrightarrow & \mathbb{P}_{\mathbf{k}}^N. \end{array}$$

The next lemma shows that the assumption that \mathcal{V} splits on \mathbb{Y} is *a priori weaker* than to say that \mathcal{V} splits on $\mathcal{Y} \times_{\mathbb{P}_{\mathbf{k}}^N} \mathrm{Spec}(\mathbb{k})$, that is on the generic section of \mathcal{N} . The upshot of this section is to prove that the ampleness of \mathcal{N} actually forces \mathcal{V} to split on X . We believe that this fact is indeed unexpected.

Lemma 4.2 *Assume that the restriction of \mathcal{V} to the geometric generic section of \mathcal{N} splits. Then there is a non-empty Zariski open subset S of $\mathbb{P}_{\mathbf{k}}^N$, and a finite cover $S' \rightarrow S$ such that $q^*\mathcal{V} \times_S S'$ splits on $\mathcal{Y}_{S'}$, and Y_s is smooth for all $s \in S$. If $\mathbf{k} = \mathbb{C}$, there is an open ball $B \subset \mathbb{P}_{\mathbb{C}}^N$ with the previous two properties.*

Proof. Let $(q^*\mathcal{V})_{\mathbb{Y}}$ be the pull-back of $q^*\mathcal{V}$ to \mathbb{Y} ; there are $\ell'_1, \dots, \ell'_r \in \mathrm{Pic}(\mathbb{Y})$ such that $(q^*\mathcal{V})_{\mathbb{Y}} = \ell'_1 \oplus \dots \oplus \ell'_r$. Since ℓ'_1, \dots, ℓ'_r are defined over an intermediate field $\mathbb{k} \hookrightarrow \mathbb{k}' \hookrightarrow \bar{\mathbb{k}}$ finitely generated and algebraic over \mathbb{k} , there is an open affine $S \subset \mathbb{P}(\Gamma(X, \mathcal{N}))$, an affine variety S' , and a finite morphism $S' \xrightarrow{\sigma} S$ such that ℓ'_1, \dots, ℓ'_r are defined over $\mathbf{k}[S']$ and $(q^*\mathcal{V})_{S'}$ splits on $\mathcal{Y}_{S'}$. After shrinking S further, Y_s is smooth for all $s \in S$, by Bertini's theorem.

If $\mathbf{k} = \mathbb{C}$, there are open balls $B' \subset S'$ and $B \subset S$ such that $\sigma : B' \rightarrow B$ is an isomorphism. Then the splitting of $(q^*\mathcal{V})_{B'}$ descends to $(q^*\mathcal{V})_B$ on \mathcal{Y}_B . \square

Henceforth we assume $\mathbf{k} = \mathbb{C}$ and consider an open ball $B \subset \mathbb{P}(\Gamma(X, \mathcal{N}))$ as above. We choose an isotypical decomposition:

$$(q^*\mathcal{V})_B = \bigoplus_{j \in J} \ell_j \otimes \mathbb{C}^{m_j}, \text{ with } \ell_j \in \mathrm{Pic}(\mathcal{Y}_B) \text{ pairwise non-isomorphic.} \quad (4.3)$$

For $(s, t) \in B \times B$, the intersection $Y_{st} := Y_s \cap Y_t$ is the zero locus of $(s, t) \in \Gamma(X, \mathcal{N}^{\oplus 2})$. Since \mathcal{N} is globally generated, Y_s and Y_t meet transversally for (s, t) in an open, dense subset $(B \times B)^\circ$. We consider the diagram:

$$\begin{array}{ccccc} & & \mathrm{Pic}(Y_s) & & \\ & \nearrow \mathrm{res}_{Y_s}^X & & \searrow \mathrm{res}_{Y_{st}}^{Y_s} & \\ \mathrm{Pic}(X) & \xrightarrow{\mathrm{res}_{Y_{st}}^X} & & \xrightarrow{\mathrm{res}_{Y_{st}}^{Y_t}} & \mathrm{Pic}(Y_{st}). \\ & \searrow \mathrm{res}_{Y_t}^X & \mathrm{Pic}(Y_t) & \nearrow \mathrm{res}_{Y_{st}}^{Y_t} & \end{array} \quad (4.4)$$

Assume that $\dim X \geq 5$. Then the Lefschetz-Sommese theorem [18] implies that all the arrows are isomorphisms, for all $(s, t) \in (B \times B)^\circ$.

Now assume that $\dim X = 4$. Since $\kappa_{Y_s} \otimes \mathcal{N} = (\kappa_X \otimes \mathcal{N}^2) \otimes \mathcal{O}_{Y_s}$ is globally generated, the Noether-Lefschetz theorem [17] implies that the (4.4) consists of isomorphisms for a dense subset of $(B \times B)^\circ$.

Lemma 4.3 *The pull-back $\text{Pic}(X) \xrightarrow{q^*} \text{Pic}(\mathcal{Y}_B)$ is an isomorphism, so*

$$(q^*\mathcal{V})_B \cong q^*\left(\bigoplus_{j \in J} \mathcal{L}_j^{\oplus m_j}\right) \otimes \mathcal{O}_{\mathcal{Y}_B}, \text{ with } \mathcal{L}_j \in \text{Pic}(X).$$

Proof. Fix $o \in B$. The composition $\text{Pic}(X) \xrightarrow{q^*} \text{Pic}(\mathcal{Y}_B) \xrightarrow{\text{res}_{Y_o}} \text{Pic}(Y_o)$ is bijective, so q^* is injective. For the surjectivity, take $\ell \in \text{Pic}(\mathcal{Y}_B)$. If $\ell_{Y_o} \cong \mathcal{O}_{Y_o}$, then

$$\{s \in B \mid \ell_{Y_s} \not\cong \mathcal{O}_{Y_s}\} = \{s \in S \mid h^0(\ell_{Y_s}) = 0\}$$

is open, by semi-continuity, so $\{s \in B \mid \ell_{Y_s} \cong \mathcal{O}_{Y_s}\}$ is closed. On the other hand, by restricting to Y_{os} , the previous discussion implies that this set is dense; thus it is the whole B . It follows that $\ell \cong \pi^*\bar{\ell}$, with $\bar{\ell} \in \text{Pic}(B)$, so $\ell \cong \mathcal{O}$. If $\ell \in \text{Pic}(\mathcal{Y})$ is arbitrary, take $\mathcal{L} \in \text{Pic}(X)$ such that $\ell_{Y_o} \cong \mathcal{L}_{Y_o}$, so $(q^*\mathcal{L}^{-1})\ell|_{Y_o}$ is trivial. \square

For all $s \in B$, let $M_s \subset J$ be the subset of maximal elements with respect to (1.3), corresponding to the splitting of $\mathcal{V} \otimes \mathcal{O}_{Y_s}$. By semi-continuity, for any $s \in B$, there is a neighbourhood $B_s \subset B$ of s such that $M_s \subset M_{s'}$ for all $s' \in B_s$. Thus there is a largest subset $M \subset J$, and an open subset $B' \subset B$ such that $M = M_s$ for all $s \in B'$.

Lemma 4.4 *Assume that $\mathbf{k} = \mathbb{C}$ and (4.1) is satisfied. Furthermore, let $B \subset \mathbb{P}_{\mathbb{C}}^N$ be a ball such that Y_s is smooth for all $s \in B$, $(q^*\mathcal{V})_B$ splits over \mathcal{Y}_B , and the set of maximal elements $M \subset J$ with respect to \prec is the same for all $s \in B$.*

We consider the (analytic) open subset $\mathcal{U} := q(\mathcal{Y}_B) \subset X$. Then there is an injective homomorphism of vector bundles $(\bigoplus_{\mu \in M} \mathcal{L}_{\mu}^{\oplus m_{\mu}}) \otimes \mathcal{O}_{\mathcal{U}} \rightarrow \mathcal{V} \otimes \mathcal{O}_{\mathcal{U}}$ whose restriction to Y_s is the natural evaluation (1.4), for all $s \in B$.

Proof. The restriction to Y_s of $\text{ev} : \bigoplus_{\mu \in M} q^*\mathcal{L}_{\mu} \otimes \pi^*\pi_*q^*(\mathcal{L}_{\mu}^{-1} \otimes \mathcal{V})_B \rightarrow (q^*\mathcal{V})_B$ is the homomorphism (1.4), for all $s \in B$. The maximality of $\mu \in M$ implies that $\pi_*q^*(\mathcal{L}_{\mu}^{-1} \otimes \mathcal{V}) \cong \mathcal{O}_B^{\oplus m_{\mu}}$ and ev is pointwise injective. We prove that, after suitable choices of bases in $\pi_*q^*(\mathcal{L}_{\mu}^{-1} \otimes \mathcal{V})$, $\mu \in M$, the homomorphism ev descends to \mathcal{U} . We will deal with each index separately, the overall basis being the direct sum of the individual ones.

Consider $\mu \in M$, and a base point $o \in B$. Then $\mathcal{V}' := \mathcal{L}_{\mu}^{-1} \otimes \mathcal{V}$ has the following properties:

- $(q^*\mathcal{V}')_B \cong \mathcal{O}_{\mathcal{Y}_B}^{\oplus m} \oplus \left(\bigoplus_{j \in J \setminus \{\mu\}} q^*(\mathcal{L}_{\mu}^{-1} \mathcal{L}_j)_B^{\oplus m_j} \right)$.
- $\pi_*(q^*\mathcal{V}')_B \cong \mathcal{O}_B^{\oplus m}$; we choose an isomorphism α_B between them.
- $\pi^*\pi_*(q^*\mathcal{V}')_B \rightarrow (q^*\mathcal{V}')_B$ is pointwise injective; let $\mathcal{T} \subset (q^*\mathcal{V}')_B$ be its image.

We choose a complement $\mathcal{W} \cong \bigoplus_{j \in J \setminus \{\mu\}} q^*(\mathcal{L}_{\mu}^{-1} \mathcal{L}_j)^{\oplus m_j}$ of \mathcal{T} in $(q^*\mathcal{V}')_B$, that is

$$(q^*\mathcal{V}')_B = \mathcal{T} \oplus \mathcal{W}. \quad (4.5)$$

The isomorphism α_B above determines the pointwise injective homomorphism

$$\alpha : \mathcal{O}_{\mathcal{Y}_B}^{\oplus m} \rightarrow (q^*\mathcal{V}')_B = \mathcal{T} \oplus \mathcal{W}$$

whose second component vanishes, as $\Gamma(\mathcal{Y}_B, \mathcal{W}) = 0$. Let $\beta : (q^*\mathcal{V}')_B \rightarrow \mathcal{O}_{\mathcal{Y}_B}^{\oplus m}$ be the left inverse of α with respect to the splitting (4.5), and note $\alpha \circ \beta|_{\mathcal{T}} = \mathbb{1}_{\mathcal{T}}$.

Claim After a suitable change of coordinates in $\mathcal{O}_{\mathcal{Y}_B}^{\oplus m}$, the homomorphisms α descends to $\mathcal{U} = q(\mathcal{Y}_B) \subset X$. Indeed, for any $s \in B$, we consider the diagram

$$\begin{array}{ccc} \mathcal{O}_{Y_{os}}^{\oplus m} & \xrightarrow{\alpha_o} & \mathcal{V}'_{Y_{os}} \\ \downarrow a_s & & \parallel \\ \mathcal{O}_{Y_{os}}^{\oplus m} & \xrightarrow{\alpha_s} & \mathcal{V}'_{Y_{os}} \\ & \nwarrow \beta_s & \end{array} \quad \text{with } a_s := \beta_s \circ \alpha_o \in \text{End}(\mathbb{C}^m).$$

Similarly, we let $a'_s := \beta_o \circ \alpha_s$. Then holds $a'_s a_s = \beta_o \alpha_s \beta_s \alpha_o = \beta_o \alpha_o = \mathbb{1}$ (for the second equality notice $\text{Im}(\alpha_o|_{Y_{os}}) = \mathcal{T}_{Y_{os}} = \text{Im}(\alpha_s|_{Y_{os}})$, and $\alpha_s \beta_s|_{\mathcal{T}} = \mathbb{1}$), and similarly $a_s a'_s = \mathbb{1}$. Thus $a_s \in \text{Gl}(m; \mathbb{C})$ for all $s \in B$, and the new trivialization $\tilde{\alpha} := \alpha \circ a$ of \mathcal{T} satisfies

$$\tilde{\alpha}_s = \tilde{\alpha}_o \text{ along } Y_{os}, \quad \forall s \in B, \quad (4.6)$$

because $\tilde{\alpha}_s|_{Y_{os}} = (\alpha_s \beta_s) \alpha_o|_{Y_{os}} = \alpha_o|_{Y_{os}} = \tilde{\alpha}_o|_{Y_{os}}$. Now we observe that, for all $s, t \in B$, the trivializations of $\mathcal{T}_{Y_{st}}$ induced by $\tilde{\alpha}$ from Y_s and Y_t , coincide. Equivalently, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_{Y_{st}}^{\oplus m} & \xrightarrow{\tilde{\alpha}_s} & \mathcal{T}_{Y_{st}} \subset \mathcal{V}'_{Y_{st}} \\ \parallel & & \parallel \\ \mathcal{O}_{Y_{st}}^{\oplus m} & \xrightarrow{\tilde{\alpha}_t} & \mathcal{T}_{Y_{st}} \subset \mathcal{V}'_{Y_{st}} \end{array} \quad \Leftrightarrow \quad \tilde{\alpha}_t^{-1} \circ \tilde{\alpha}_s|_{Y_{st}} = \mathbb{1} \in \text{Gl}(r; \mathbb{C}). \quad (4.7)$$

Indeed, the triple intersection Y_{ost} —the zero locus of $(o, s, t) \in \Gamma(X, \mathcal{N}^{\oplus 3})$ —is a non-empty, connected subscheme of X , as $\dim X - 3\nu \geq 1$. Hence it is enough to prove that the restriction of (4.7) to Y_{ost} is the identity. After restricting (4.6) to Y_{ost} , we deduce

$$\tilde{\alpha}_s|_{Y_{ost}} = \tilde{\alpha}_o|_{Y_{ost}} = \tilde{\alpha}_t|_{Y_{ost}} \quad \Rightarrow \quad \tilde{\alpha}_t^{-1} \circ \tilde{\alpha}_s|_{Y_{ost}} = \mathbb{1}.$$

Finally, we conclude that the trivialization $\tilde{\alpha}$ of $\pi_* q^*(\mathcal{L}_\mu^{-1} \otimes \mathcal{V})$ descends to \mathcal{U} , as announced. Indeed, we define $\bar{\alpha} : \mathcal{O}_{\mathcal{U}}^{\oplus m} \rightarrow \mathcal{V} \otimes \mathcal{O}_{\mathcal{U}}$, $\bar{\alpha}(x) := \tilde{\alpha}_s(x)$ for some $s \in B$ such that $x \in Y_s$. The diagram (4.7) implies that $\bar{\alpha}(x)$ is independent of $s \in B$ with $s(x) = 0$. \square

Remark 4.5 Let B be as in 4.2. The proofs of the Lemmas 4.3 and 4.4 require only the following consequences of (4.1):

- (i) For all $s \in B$, $\text{Pic}(X) \rightarrow \text{Pic}(Y_s)$ is an isomorphism;
- (ii) For all $s \in B$, there is $B_s \subset B$ dense, such that the intersection Y_{st} , $t \in B_s$, is transverse and the diagram (4.4) consists of isomorphisms;
- (iii) Y_o, Y_{st}, Y_{ost} are connected, for all $o, s, t \in B$.

This observation will be used in the proof of the Theorem 6.4.

Lemma 4.6 *Let the situation be as in the Lemma 4.4. Then \mathcal{V} is obtained as a successive extension of line bundles on X .*

Proof. Let $(q^* \mathcal{V})_B = \bigoplus_{j \in J} q^* \mathcal{L}_j \otimes \mathbb{C}^{m_j}$ be an isotypical decomposition, with $\mathcal{L}_j \in \text{Pic}(X)$. First we prove the lemma over \mathcal{U} , by induction on the cardinality of J . For $|J| = 1$, we have $(q^*(\mathcal{L}^{-1} \otimes \mathcal{V}))_B \cong \mathcal{O}_{\mathcal{Y}_B}^{\oplus m}$ for some $\mathcal{L} \in \text{Pic}(X)$. The Lemma 4.4 implies $\mathcal{V}_{\mathcal{U}} \cong \mathcal{L} \otimes \mathcal{O}_{\mathcal{U}}^{\oplus m}$.

Now suppose that the lemma holds for $|J| \leq n$, and prove it for $|J| = n + 1$. For the maximal elements $M \subset J$, there is a pointwise injective homomorphism $\bigoplus_{\mu \in M} \mathcal{L}_\mu \otimes \mathcal{O}_\mathcal{U}^{\oplus m_\mu} \rightarrow \mathcal{V}_\mathcal{U}$.

Its cokernel $\mathcal{W}_\mathcal{U}$ is locally free over \mathcal{U} and $q^*\mathcal{W}_\mathcal{U} \cong \bigoplus_{j \in J \setminus M} q^*\mathcal{L}_j^{\oplus m_j}$. By the induction hypothesis, $\mathcal{W}_\mathcal{U}$ is obtained by successive extensions of \mathcal{L}_j , $j \in J \setminus M$, so the same holds for $\mathcal{V}_\mathcal{U}$.

It remains to prove that \mathcal{V} itself is a successive extension of line bundles on X . This follows by repeatedly applying the Corollary 2.3. Indeed, each of the successive extensions involved in $\mathcal{V}_\mathcal{U}$ uniquely extends to the whole X , as \mathcal{U} is an open neighbourhood of Y_o , $o \in B$. But $\mathcal{V}_\mathcal{U}$ is the result of this process over \mathcal{U} , and \mathcal{V} is already defined on the whole X , so the uniqueness part of the Corollary 2.3 yields the conclusion. \square

Theorem 4.7 *Let X be an irreducible, smooth, projective \mathbf{k} -variety, and \mathcal{N} a globally generated, ample vector bundle on it satisfying (4.1). We assume that the restriction of \mathcal{V} to the geometric generic section \mathbb{Y} of \mathcal{N} splits. Then the following statements hold:*

- (i) *If \mathbf{k} is uncountable, \mathcal{V} is a successive extension of line bundles on X .*
- (ii) *Assume that \mathbf{k} is arbitrary and either one of the following two conditions is fulfilled:*
 - (H1) *$H^1(X, \mathcal{L}) = 0$, for all $\mathcal{L} \in \text{Pic}(X)$;*
 - (SS) *$\Gamma(\mathbb{Y}, \mathcal{E}nd(\mathcal{V}_\mathbb{Y}))$ is a semi-simple, finite dimensional algebra.**Then \mathcal{V} is a split vector bundle on X .*

Proof. The proof is done in two steps.

Case $\mathbf{k} = \mathbb{C}$. Let $B \subset \mathbb{P}(\Gamma(X, \mathcal{N}))$ be as in the Lemma 4.4, and decompose $(q^*\mathcal{V})_B = \bigoplus_{j \in J} q^*\mathcal{L}_j \otimes \mathbb{C}^{m_j}$. The Lemma 4.6 says that \mathcal{V} is a successive extension of \mathcal{L}_j , $j \in J$. Now assume

that either (H1) or (SS) is satisfied. On one hand, if $H^1(X, \mathcal{L}) = 0$ for all $\mathcal{L} \in \text{Pic}(X)$, then any extension of line bundles is trivial, so \mathcal{V} is isomorphic to $\bigoplus_{j \in J} \mathcal{L}_j^{\oplus m_j}$. On the other hand,

$\Gamma(\mathbb{Y}, \mathcal{E}_\mathbb{Y})$ is semi-simple if and only if $\Gamma(\mathbb{Y}, \ell_i^{-1}\ell_j) = 0 \Leftrightarrow \Gamma(Y_s, \mathcal{L}_i^{-1}\mathcal{L}_j) = 0$, $\forall i \neq j \forall s \in B$. In this case all the elements of J are maximal with respect to (1.3), and the conclusion follows from the Lemma 4.4.

Case \mathbf{k} arbitrary. Let $\mathbf{k}_0 \subset \mathbf{k} \cap \mathbb{C}$ be a countable, algebraically closed field, such that $X, \mathcal{N}, \mathcal{V}$ are defined over \mathbf{k}_0 , and let $X_0, \mathcal{N}_0, \mathcal{V}_0$ be the corresponding objects. Then the geometric generic fibres fit into the Cartesian diagram

$$\begin{array}{ccc} \mathcal{V}_{\bar{\mathbf{k}}} & \xrightarrow{\psi} & \mathcal{V}_{\bar{\mathbf{k}}_0} \\ \downarrow & & \downarrow \pi \\ \text{Spec}(\bar{\mathbf{k}}) & \longrightarrow & \text{Spec}(\bar{\mathbf{k}}_0), \end{array}$$

and $(q^*\mathcal{V})_{\bar{\mathbf{k}}_0}$ splits by the Proposition 1.7, so $(q^*\mathcal{V})_{\bar{\mathbf{k}}_0} \times_{\mathbf{k}_0} \mathbb{C}$ splits too. But this latter is the restriction of $\mathcal{V}_\mathbb{C} := \mathcal{V} \times_{\mathbf{k}_0} \mathbb{C}$ to the zero locus of the geometric generic section of $\mathcal{N}_\mathbb{C}$, hence $\mathcal{V}_\mathbb{C}$ is a successive extension of line bundles \mathcal{L}_j on $X_\mathbb{C}$, $j \in J$, by the previous step.

There is an intermediate field $\mathbf{k}_0 \hookrightarrow \mathbf{k}_1 \hookrightarrow \mathbb{C}$ of finite type over \mathbf{k}_0 , such that \mathcal{L}_j , $j \in J$, are defined over \mathbf{k}_1 , thus $\mathcal{V}_0 \times_{\mathbf{k}_0} \mathbf{k}_1$ is a successive extension of line bundles on $X_0 \times_{\mathbf{k}_0} \mathbf{k}_1$.

On one hand, if \mathbf{k} is uncountable, the transcendence degree of \mathbf{k} over \mathbf{k}_0 is infinite because \mathbf{k}_0 is countable. Hence we can realize \mathbf{k}_1 as a sub-field of \mathbf{k} , and the conclusion follows by base change.

On the other hand, if either (H1) or (SS) is fulfilled (over \mathbf{k}), then the same holds over \mathbf{k}_0 and \mathbb{C} , so $\mathcal{V}_{\mathbb{C}}$ splits. By applying 1.7 once more, we deduce the splitting of \mathcal{V}_0 on X_0 and of \mathcal{V} on X . \square

Remark 4.8 (i) If $\Gamma(\mathcal{E}_{\mathbb{Y}})$ is not semi-simple, then \mathcal{V} is a successive extension of line bundles on X , and we don't know whether \mathcal{V} actually splits. The difficulty is that the unipotent automorphisms of $\mathcal{V}_{\mathbb{Y}}$ act non-trivially on the isotypical decompositions of $\mathcal{V}_{\mathbb{Y}}$.

(ii) We cannot decide the optimality of the factor $1/3$ in (4.1). The following example illustrates why the triple intersections Y_{ost} , $o, s, t \in \Gamma(X, \mathcal{N})$, are assumed non-empty and connected. Let $\mathcal{V} = \mathcal{T}_{\mathbb{P}_{\mathbb{C}}^2}$ be the tangent bundle of $X = \mathbb{P}_{\mathbb{C}}^2$. It is a non-split, uniform vector bundle of rank two, and its restriction to any line $Y \subset \mathbb{P}_{\mathbb{C}}^2$ is isomorphic to $\mathcal{O}_Y(2) \oplus \mathcal{O}_Y(1)$. The incidence variety \mathcal{Y} is the variety of full flags in \mathbb{C}^3 , and we have the diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{q} & \mathbb{P}_{\mathbb{C}}^2 \\ \pi \downarrow \mathbb{P}_{\mathbb{C}}^1\text{-fibration} & & \\ |\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(1)| & \cong & \mathbb{P}_{\mathbb{C}}^2 \end{array}$$

The geometric generic fibre \mathbb{Y} of π is isomorphic to the projective line defined over the algebraic closure of the quotient field of $\mathbb{P}_{\mathbb{C}}^2$, so $q^*\mathcal{T}_{\mathbb{P}_{\mathbb{C}}^2}$ splits on \mathbb{Y} , and there is a ball $B \subset |\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(1)|$ such that $(q^*\mathcal{T}_{\mathbb{P}_{\mathbb{C}}^2})|_{\pi^{-1}(B)}$ splits. However, this splitting does not descend to $q(\pi^{-1}(B)) \subset \mathbb{P}_{\mathbb{C}}^2$, for no such B . Otherwise, the Proposition 2.5 would imply that $\mathcal{T}_{\mathbb{P}_{\mathbb{C}}^2}$ splits, a contradiction.

For *arbitrary* varieties defined over *uncountable* ground fields (*e.g.* \mathbb{C}), it is enough to check the splitting of the restriction to a *single* sufficiently general ample subvariety.

Theorem 4.9 *Assume that (4.1) is satisfied, and \mathbf{k} is uncountable. Let $\mathbf{k}_0 \subset \mathbf{k}$ be a countable, algebraically closed sub-field such that $X, \mathcal{N}, \mathcal{V}$ are defined over \mathbf{k}_0 . Consider a regular section $s \in \Gamma(X, \mathcal{N})$ with the following properties:*

- \mathcal{V}_{Y_s} is split;
- in some affine chart induced from $\mathbb{P}_{\mathbf{k}_0}^N$, the coordinates of $[s] \in \mathbb{P}_{\mathbf{k}}^N$ are algebraically independent over \mathbf{k}_0 .

Assume furthermore that either one of the following two conditions is satisfied:

- (H1) $H^1(X, \mathcal{L}) = 0$ for all $\mathcal{L} \in \text{Pic}(X)$;
- (SS) $\Gamma(Y_s, \mathcal{E}nd(\mathcal{V}_{Y_s}))$ is a semi-simple, finite dimensional algebra.

Then the vector bundle \mathcal{V} splits into a direct sum of line bundles on X .

As \mathbf{k}_0 is countable, the points $[s] \in \mathbb{P}_{\mathbf{k}}^N$ with the previous properties lie in the complement of a countable union of proper subvarieties of $\mathbb{P}_{\mathbf{k}}^N$, so we can reformulate as follows:

If \mathbf{k} is uncountable, and the restriction to the zero locus of a very general section of \mathcal{N} splits, then the vector bundle \mathcal{V} does the same.

Proof. Let (c_1, \dots, c_N) be the coordinates of $[s]$ in the affine chart $c_0 \neq 0$ on $\mathbb{P}_{\mathbf{k}}^N$. By assumption, they are algebraically independent over \mathbf{k}_0 , which implies

$$\mathbf{k}_0 \subset \mathbb{k}_0 := \mathbf{k}_0(\xi_1, \dots, \xi_N) \cong \mathbf{k}_0(c_1, \dots, c_N) \subset \mathbf{k} \xrightarrow{\mathbf{k} \text{ alg. closed}} \bar{\mathbb{k}}_0 \subset \mathbf{k}.$$

(Here ξ_1, \dots, ξ_N are indeterminates.) Therefore the closed point $[s] \in \mathbb{P}_{\mathbf{k}}^N$ maps to the generic point of $\mathbb{P}_{\mathbf{k}_0}^N$. Moreover, as \mathbf{k}_0 is countable, it can be realized as a sub-field of \mathbb{C} . We consider

the following diagram:

$$\begin{array}{ccccccc}
 & & X & \xrightarrow{b} & X_{\bar{\mathbf{k}}_0} & \xrightarrow{\quad} & X_{\mathbf{k}_0} \\
 & \nearrow q & \downarrow & \nearrow \bar{q}_0 & \downarrow & \nearrow q_0 & \downarrow \\
 Y_s & \xrightarrow{\quad} & \mathcal{Y}_{\bar{\mathbf{k}}_0} & \xrightarrow{\quad} & \mathcal{Y}_{\mathbf{k}_0} & \xrightarrow{\quad} & \mathcal{Y}_{\mathbf{k}_0} \\
 \downarrow \pi & \searrow & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \text{Spec}(\mathbf{k}) & \xrightarrow{\quad} & \text{Spec}(\bar{\mathbf{k}}_0) & \xrightarrow{\quad} & \text{Spec}(\mathbf{k}_0) \\
 \downarrow [s] & \nearrow & \downarrow \bar{\pi}_0 & \nearrow & \downarrow \pi_0 & \nearrow & \downarrow \\
 & \mathbb{P}_{\bar{\mathbf{k}}_0}^N & \xrightarrow{\quad} & \mathbb{P}_{\mathbf{k}_0}^N & \xrightarrow{\quad} & \mathbb{P}_{\mathbf{k}_0}^N
 \end{array}$$

Now we focus on the diagonal Cartesian rectangle with dotted sides. Our hypothesis is that \mathcal{V}_{Y_s} splits. Since both $\bar{\mathbf{k}}_0$ and \mathbf{k} are algebraically closed, the Proposition 1.7 implies that $(q_0^* \mathcal{V})_{\bar{\mathbf{k}}_0}$ on $\mathcal{Y}_{\bar{\mathbf{k}}_0}$ splits, so $(q_0^* \mathcal{V})_{\bar{\mathbf{k}}_0} \times_{\mathbf{k}_0} \mathbb{C}$ splits too. The Theorem 4.7 implies that $\mathcal{V}_{\mathbb{C}}$ splits and we conclude that the initial \mathcal{V} is split, by the Proposition 1.7. \square

The theorem can be used to create a variety of applications. It is not clear how to handle the following examples by using different methods.

Example 4.10 (i) (Compare with 3.3(i)) Consider $X \hookrightarrow \text{Gr}(n; \mathbb{C}^{n+\nu})$ of codimension c , with $n \geq 4, \nu \geq 2, c \leq (n-3)\nu - 1$, and let \mathcal{G} be the universal quotient. If X satisfies the condition (H1), then the splitting of \mathcal{V} along the zero locus of a very general section of $\mathcal{N} := \mathcal{G}(1)_X$ implies its global splitting.

(ii) Let \mathcal{V} be a vector bundle on $X := \mathbb{P}_{\mathbb{C}}^m \times \mathbb{P}_{\mathbb{C}}^n$, with $n \geq 2m+1$ and $m \geq 2$. We consider the ample vector bundle $\mathcal{N} := \mathcal{T}_{\mathbb{P}_{\mathbb{C}}^m} \boxtimes \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^n}(1)$, and a very general section $s \in \Gamma(X, \mathcal{N})$. Then \mathcal{V} splits if and only if its restriction to Y_s does so. (Note that $Y_s \subset X$ has codimension m , and $Y_s \rightarrow \mathbb{P}_{\mathbb{C}}^n$ is a $(m+1)$ -sheeted ramified covering.)

5. SPLITTING ALONG DIVISORS

Here we elaborate on the case where \mathcal{N} is an ample line bundle and Y is a divisor. Throughout $\dim_{\mathbf{k}} X \geq 3$, and $\mathcal{O}_X(1)$ is an ample line bundle on X .

Lemma 5.1 *Let $D \in |\mathcal{O}_X(m)|$, with $m \geq 1$, be a divisor such that*

$$H^1(D, \mathcal{E}_D(-a)) = 0, \quad \forall a \geq c. \quad (\text{Recall that } \mathcal{E} = \mathcal{E}nd(\mathcal{V}).) \quad (5.1)$$

Then hold:

- (i) *The cohomology group $H^1(X, \mathcal{E}(-a))$ vanishes for all $a \geq c$.*
- (ii) *Assume moreover that $m \geq c$ and \mathcal{V}_D splits. Then \mathcal{V} splits too.*

Proof. (i) The Serre vanishing implies:

$$a_0 := \min\{a \geq c \mid H^1(X, \mathcal{E}(-a)) = 0, \forall a \geq a\} < \infty.$$

If $a_0 \geq c+1$, the exact sequence $0 \rightarrow \mathcal{O}_X(-m) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$ yields

$$\dots \rightarrow H^1(\mathcal{E}(-m-a_0+1)) \rightarrow H^1(\mathcal{E}(-a_0+1)) \rightarrow H^1(\mathcal{E}_D(-a_0+1)) \rightarrow \dots,$$

with $-m-a_0+1 \leq -a_0$, $a_0-1 \geq c$; the first and last terms vanish, so the middle term vanishes too, which contradicts the minimality of a_0 .

(ii) As $m \geq c$, the first step implies that $\text{res}_D : \Gamma(X, \mathcal{E}) \rightarrow \Gamma(D, \mathcal{E}_D)$ is surjective. Hence \mathcal{V} splits, by the Lemma 1.6(ii). \square

Theorem 5.2 *Assume that either*

(i) $D \in |\mathcal{O}_X(m)|$ *is normal, and* $\mathcal{E}_D(m)$ *is ample (e.g. $\mathcal{E}(m)$ is ample),*
or (ii) $H^1(D, \ell) = 0, \forall \ell \in \text{Pic}(D)$. (See [3, Proposition 4.13, Corollary 4.14].)
Then \mathcal{V} splits if and only if \mathcal{V}_D splits.

The criterion implies that \mathcal{V} splits if and only if its restriction to a complete intersection surface in X of sufficiently high degree splits.

Proof. (i) By hypothesis $\mathcal{V}_D = \bigoplus_{j=1}^r \ell_j$ with $\ell_j \in \text{Pic}(D)$. As $\mathcal{E}_D(m)$ is ample, $\ell_i^{-1} \ell_j \otimes \mathcal{O}_D(m+a)$ is ample, for all i, j and $a \geq 0$. The Kodaira vanishing theorem [15] yields $H^1(\mathcal{E}_D(-m-a)) = 0, \forall a \geq 0$, which is the condition (5.1).

(ii) Since \mathcal{V}_D splits, $H^1(\mathcal{E}_D(-a)) = 0$ for all $a \geq 0$; we conclude as before. \square

Varieties enjoying additional properties admit stronger splitting criteria.

Definition 5.3 Let X be a scheme and $h \geq 1$ be an integer. We say that X is an h -*splitting scheme* if $H^1(X, \mathcal{L}) = \dots = H^h(X, \mathcal{L}) = 0$ for all line bundles $\mathcal{L} \rightarrow X$. The cases $h = 1, 2$ respectively correspond to the notions of *splitting* and *Horrocks scheme* in [3].

If $(X, \mathcal{O}_X(1))$ is d -dimensional, arithmetically Cohen-Macaulay, with $\text{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$, then X is $(d-1)$ -splitting. (This follows directly from the definition and the Kodaira vanishing.) Examples include Fano varieties with cyclic Picard groups (e.g. homogeneous spaces G/P , with $P \subset G$ a maximal parabolic subgroup), and smooth (resp. very general) complete intersections of dimension $d \geq 4$ (resp. $d \geq 3$) in them.

The next result generalizes [3, Corollary 4.14] because we allow 1- rather than 2-splitting varieties.

Theorem 5.4 *Assume \mathbf{k} is uncountable, $\mathcal{O}_X(m)$ is globally generated; take $D \in |\mathcal{O}_X(m)|$ very general. (So D is smooth). In either one of the following situations, \mathcal{V} splits if and only if its restriction \mathcal{V}_D splits:*

- (i) X is 2-splitting, $\dim_{\mathbf{k}}(X) = 3$, and $\kappa_X(m)$ is generated by global sections. (Here κ_X stands for the canonical line bundle.)
- (ii) X is 1-splitting, $\dim_{\mathbf{k}}(X) = 4$, and $\kappa_X(2m)$ is generated by global sections.
- (iii) X is 1-splitting, $\dim_{\mathbf{k}}(X) \geq 5$.

Proof. (i) The Noether-Lefschetz theorem [17] states that $\text{Pic}(X) \rightarrow \text{Pic}(D)$ is an isomorphism. Thus for any $\ell \in \text{Pic}(D)$ there is $\mathcal{L} \in \text{Pic}(X)$ such that $\mathcal{L}_D = \ell$. The long exact sequence in cohomology associated to $0 \rightarrow \mathcal{L}(-m) \rightarrow \mathcal{L} \rightarrow \ell \rightarrow 0$ yields $H^1(D, \ell) = 0$. Hence $H^1(D, \mathcal{E}_D(-a)) = 0, \forall a \geq 0$ because \mathcal{E}_D splits, by hypothesis. Now apply the Lemma 5.1(ii).
(ii), (iii) The statements are particular cases of the Theorem 4.9. \square

An interesting application for vector bundles on the Grassmannian, which combines our results obtained so far, is given in 6.3.

Remark 5.5 The same arguments work in positive characteristics. Assume that $\text{char}(\mathbf{k}) > \dim_{\mathbf{k}} X \geq 4$, the pair $(X, \mathcal{O}_X(1))$ admits a $W_2(\mathbf{k})$ -lifting, and X is 2-splitting. Then \mathcal{V} splits if and only if \mathcal{V}_D splits.

6. SPLITTING OF VECTOR BUNDLES ON GRASSMANNIANS (THE AMPLENESS OF \mathcal{N} IS NECESSARY?)

Throughout this article we restricted \mathcal{V} to ‘test subvarieties’ which are zero loci of regular sections in an *ample* vector bundle \mathcal{N} . However, our strategy to prove the splitting of \mathcal{V} (cf. page 4) makes sense without this restriction, and is natural to ask whether the hypothesis can be weakened. In the general setting of the Section 2, this does not seem possible, because the ampleness hypothesis is used to deduce the vanishing of certain cohomology groups. However, the answer to the question appears to be affirmative if \mathcal{N} is globally generated and one has *a priori* information about the zero loci of its regular sections. We illustrate our statement with two concrete examples.

6.1. The case of the Grassmannian.

Theorem 6.1 *A vector bundle \mathcal{V} on $X := \text{Gr}(e; \mathbf{k}^d)$, with $e, d - e \geq 2$, is split if and only if its restriction to $Y := \{U \in \text{Gr}(e; \mathbf{k}^d) \mid \mathbf{k}^{e-2} \subset U \subset \mathbf{k}^{e+2}\} \cong \text{Gr}(2; \mathbf{k}^4)$ is so.*

Cohomological splitting criteria for vector bundles on Grassmannians have been obtained in [16, 13, 1]. However, they involve many conditions. We believe that our result is interesting for its simplicity: it reduces the problem of splitting on the Grassmannian, which is a high dimensional object, to a 3-dimensional quadric $Q_3 \in \mathbb{P}^4$. (See the Remark 6.3 for the reduction from $\text{Gr}(2; 4)$ to Q_3 .)

For $e = 2, d = 4$ there is nothing to prove. For $e = 2, d \geq 5$, we use the duality $\text{Gr}(e; \mathbf{k}^d) \cong \text{Gr}(d - e; \mathbf{k}^d)$, so we can write $d = \nu + n + 1$ and $e = n + 1$, with $\nu, n \geq 2$. The theorem is obtained by repeatedly applying the following:

Proposition 6.2 *A vector bundle \mathcal{V} on $X = \text{Gr}(n + 1; W)$, with $W \cong \mathbf{k}^{\nu+n+1}$, $\nu, n \geq 2$, splits if and only if its restriction to some smaller Grassmannian $Y \cong \text{Gr}(n; \nu + n)$ contained in X splits.*

The universal quotient $W \otimes \mathcal{O}_X \xrightarrow{\beta} \mathcal{N}$ induces an isomorphism $W \xrightarrow{\cong} \Gamma(\mathcal{N})$, and $s \in W \setminus \{0\}$ determines a section in \mathcal{N} whose zero locus is the ‘smaller’ Grassmannian $Y := \text{Gr}(n; W_s)$, with $W_s := W / \langle s \rangle$.

We remark that \mathcal{N} is *not ample* on X because its restriction to any (straight) line $l \subset X$ is isomorphic to $\mathcal{O}_l^{\nu-1} \oplus \mathcal{O}_l(1)$.

Proof. We follow the strategy described on the page 4.

Claim 1 For any vector bundle \mathcal{F} on X holds:

- $H^1(X, \mathcal{F} \otimes \mathcal{I}_Y^m) = 0$, so $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(Y_m, \mathcal{F}_{Y_m})$ is surjective, $\forall m \gg 0$;
 - the cohomological dimension $\text{cd}(X \setminus Y) \leq \dim X - (n + 1)$.
- (6.1)

Let $\tilde{X} := \text{Bl}_Y(X)$ be the blow-up of X along Y and $\pi: \tilde{X} \rightarrow X$ be the projection; we denote the exceptional divisor by $E = \mathbb{P}(\mathcal{N}_Y) \subset \tilde{X}$. As Y is the zero locus of $s \in \Gamma(\mathcal{N})$, we have $\tilde{X} \subset \mathbb{P}(\mathcal{N})$, and holds:

- (i) $\mathcal{O}_{\tilde{X}}(-E) = \mathcal{O}_{\mathbb{P}(\mathcal{N})}(1)|_{\tilde{X}}$ is π -relatively ample;
- (ii) $H^1(X, \mathcal{F} \otimes \mathcal{I}_Y^m) = H^1(\tilde{X}, \pi^* \mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(-mE))$, $\forall m \geq 1$.

Furthermore, as $\mathcal{N} \cong \bigwedge^{\nu-1} \mathcal{N}^\vee \otimes \det(\mathcal{N})$ and \mathcal{N} is globally generated, we have

$$\mathbb{P}(\mathcal{N}) \cong \mathbb{P}\left(\bigwedge^{\nu-1} \mathcal{N}^\vee \otimes \det(\mathcal{N})\right) \subset \mathbb{P}\left(\bigwedge^{\nu-1} W^\vee \otimes \mathcal{O}_X(1)\right) \cong X \times \mathbb{P},$$

where $\mathbb{P} := \mathbb{P}(\bigwedge^{\nu-1} W^\vee)$, and $\mathcal{O}_{\mathbb{P}(\mathcal{N})}(1) = (\mathcal{O}_X(-1) \boxtimes \mathcal{O}_{\mathbb{P}}(1))|_{\mathbb{P}(\mathcal{N})}$.

Pointwise, the morphism $\mathbb{P}(\mathcal{N}) \rightarrow \mathbb{P}$ is defined by

$$(x, \langle e_x \rangle) \mapsto \det(\mathcal{N}_x / \langle e_x \rangle)^\vee \subset \bigwedge^{\nu-1} \mathcal{N}_x^\vee \subset \bigwedge^{\nu-1} W^\vee, \quad (6.2)$$

where $\langle e_x \rangle$ stands for the line generated by $e_x \in \mathcal{N}_x$. Its restriction $q : \tilde{X} \rightarrow \mathbb{P}$ to \tilde{X} corresponds to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{s} & W \otimes \mathcal{O}_X & \longrightarrow & W_s \otimes \mathcal{O}_X \longrightarrow 0 \\ & & \parallel & & \downarrow \beta & & \downarrow \\ & & \mathcal{O}_X & \xrightarrow{\beta s} & \mathcal{N} & \longrightarrow & \mathcal{N} / \langle \beta s \rangle \longrightarrow 0. \end{array}$$

The homomorphism βs is injective precisely over $X \setminus Y$ and (6.2) shows that q is the desingularization of the rational map

$$\mathrm{Gr}(n+1; W) \dashrightarrow \mathrm{Gr}(n+1; W_s), \quad [U \subset W] \mapsto [(U + \langle s \rangle) / \langle s \rangle \subset W_s], \quad (6.3)$$

followed by the usual Plücker embedding of $\mathrm{Gr}(n+1; W_s)$.

For an arbitrary vector bundle $\tilde{\mathcal{F}}$ on \tilde{X} , we have $R^j q_*(\tilde{\mathcal{F}} \otimes \pi^* \mathcal{O}_X(m)) = 0$, for all $j > 0$ and $m \gg 0$, so the Leray spectral sequence yields

$$\begin{aligned} & H^{\dim X - 1}(\tilde{X}, \tilde{\mathcal{F}} \otimes \pi^* \mathcal{O}_X(m) \otimes q^* \mathcal{O}_{\mathbb{P}}(-m)) \\ &= H^{\dim X - 1}(\mathrm{Gr}(n+1; W_s), q_*(\tilde{\mathcal{F}} \otimes \pi^* \mathcal{O}_X(m)) \otimes \mathcal{O}_{\mathbb{P}}(-m)) = 0. \end{aligned} \quad (6.4)$$

The last equality holds because $\dim X - 1 > \dim(\mathrm{Gr}(n+1; W_s))$. The Serre duality implies $H^1(X, \mathcal{F} \otimes \mathcal{I}_Y^m) = H^1(\tilde{X}, \pi^* \mathcal{F}(-m) \otimes q^* \mathcal{O}_{\mathbb{P}}(m)) \stackrel{(6.4)}{=} 0$, for $m \gg 0$.

It remains to estimate the cohomological dimension of $X \setminus Y$. For this, we observe that the morphism $q : X \setminus Y \rightarrow \mathrm{Gr}(n+1; W_s)$ is affine (cf. (6.3)), so $\mathrm{cd}(X \setminus Y) \leq \dim \mathrm{Gr}(n+1; W_s) = \dim X - (n+1)$.

Claim 2 For any line bundle ℓ on Y and $m \geq 1$ holds

$$H^1(Y, \mathrm{Sym}^m(\mathcal{N}_Y^\vee) \otimes \ell) = 0 \quad (\text{cf. 2.4(ii)}).$$

Note that $\det(\mathcal{N}_Y) = \mathcal{O}_Y(1)$ generates $\mathrm{Pic}(Y)$, so $\ell = \mathcal{O}_Y(k)$ for some $k \in \mathbb{Z}$. The cohomology group above can be computed on $F := \mathbb{P}(\mathcal{N}_Y^\vee) \xrightarrow{f} Y$:

$$H^1(Y, \mathrm{Sym}^m \mathcal{N}_Y^\vee \otimes \mathcal{O}_Y(k)) \cong H^\nu(F, \underbrace{\mathcal{O}_f(-\nu - m) \otimes \mathcal{O}_Y(k+1)}_{=: \mathcal{L}}). \quad (6.5)$$

Also, F is isomorphic to the (homogeneous) variety of partial flags

$$0 \subset U_n \subset U_{\nu+n-1} \subset W_s^{\nu+n},$$

where the indices indicate the dimensions. The vanishing of (6.5) follows from Bott's theorem [5, 7]. We follow [19, Chapter 4] which treats in detail the flag varieties for the general linear group. Let $\mathcal{U}_n, \mathcal{U}_{\nu+n-1}$ be the tautological bundles on F of indicated ranks. Then

$$\mathcal{O}_f(1) \cong (W_s \otimes \mathcal{O}_Y) / \mathcal{U}_{\nu+n-1}, \quad f^* \mathcal{O}_Y(1) \cong \det(\mathcal{U}_n)^{-1},$$

so \mathcal{L} corresponds to the weight $a = (\underbrace{k+1, \dots, k+1}_{n \text{ times}}, \underbrace{0, \dots, 0}_{\nu-1 \text{ times}}, \nu+m) \in \mathbb{Z}^{\nu+n}$ (cf. [19, pp.

112]). We denote $\rho := (\nu+n-1, \dots, 0)$. Bott's theorem says that $H^\nu(F, \mathcal{L}) = 0$ if and only if either one of the following two cases occur:

- (i) $a + \rho$ is singular—that is, it contains two identical entries;
- (ii) $a + \rho$ is non-singular and the number of strict order inversions (of the decreasing order) in $a + \rho$ is different of ν .

We claim that, for all $m \geq 1$, $a + \rho = (\nu + k + n, \dots, \nu + k + 1, \nu - 1, \dots, 1, \nu + m)$, fulfils one of the two conditions above. Note that \mathcal{L}^{-1} is ample for $k + 1 < 0$, so (5.1) vanishes by Kodaira's theorem. For $k + 1 \geq 0$, the first $\nu + n - 1$ terms of $a + \rho$ are strictly decreasing, hence only the last term can contribute to a strict inversion. Actually, $\nu + m > \nu - 1$, so we do have $\nu - 1$ inversions. There is exactly one more strict inversion precisely for $\nu + m = (\nu + k + 1) + 1$, that is $m = k + 2$. But in this case $a + \rho$ is singular since $n \geq 2$, by hypothesis.

Claim 3 Assume that \mathcal{V}_Y splits. Then $\Gamma(Y_m, \mathcal{E}_{Y_m}) \rightarrow \Gamma(Y, \mathcal{E}_Y)$ is surjective, $\forall m \geq 1$. Indeed, we tensor by \mathcal{E}_Y the exact sequence:

$$0 \rightarrow \mathrm{Sym}^m \mathcal{N}_Y^\vee \cong \mathcal{I}_Y^m / \mathcal{I}_Y^{m+1} \rightarrow \mathcal{O}_{Y_m} \rightarrow \mathcal{O}_{Y_{m-1}} \rightarrow 0.$$

Since \mathcal{E}_Y is a direct sum of line bundles, we apply the Claim 2. \square

Remark 6.3 The results obtained in the Section 5 allow to probe the splitting of the vector bundle \mathcal{V} on even lower dimensional subvarieties. Indeed, the image of $\mathrm{Gr}(2; 4)$ by the Plücker embedding is the smooth 4-dimensional quadric $Q_4 \subset \mathbb{P}^5$. Let $Q_3 \subset Q_4$ be an arbitrary smooth hyperplane section, and $S \subset \mathbb{P}^4$ be a very general intersection of Q_3 with a quartic in \mathbb{P}^4 . (Observe that $S \subset Q_4$ is a surface with $\mathrm{Pic}(S) = \mathbb{Z} \cdot \mathcal{O}_S(1)$, and $\kappa_S = \mathcal{O}_S(1)$.) The Theorems 5.2(ii) and 5.4(i)(a) respectively imply that \mathcal{V} splits on Q_4 if and only if either \mathcal{V}_{Q_3} or \mathcal{V}_S splits. (In the latter case we require \mathbf{k} to be uncountable.)

6.2. The case of partial flag varieties. Here we deduce a splitting criterion for vector bundles on partial flag varieties. To the author's knowledge, there are no previously known results in this case. First we introduce the notation. Let

$$d_\bullet := (0 = d_0 < d_1 < \dots < d_t < d_{t+1})$$

be a strictly increasing sequence of integers, and $\nu_j := d_{t+1} - d_j$, for $j = 0, \dots, t$. For shorthand, we denote $2_\bullet := (0 < 2 < \dots < 2t < 2(t+1))$.

If $d'_\bullet = (0 = d'_0 < d'_1 < \dots < d'_t < d'_{t+1})$ is another sequence, we write:

$$d'_\bullet \leq d_\bullet \Leftrightarrow d'_j - d'_{j-1} \leq d_j - d_{j-1}, \forall 1 \leq j \leq t+1. \quad (6.6)$$

We consider the partial flag variety

$$F_{d_\bullet} = \mathrm{Fl}(d_1, \dots, d_t; d_{t+1}) := \{U_\bullet = (0 \subset U_{d_1} \subset \dots \subset U_{d_t} \subset W := \mathbf{k}^{d_{t+1}})\},$$

where the indices indicate the dimensions of the vector spaces. For $d'_\bullet \leq d_\bullet$, a choice of subspaces $\mathbf{k}^{d_1-d'_1} \subset \dots \subset \mathbf{k}^{d_{t+1}-d'_{t+1}} \subset \mathbf{k}^{d_{t+1}}$ yields the embedding:

$$\begin{aligned} \iota : F_{d'_\bullet} &\rightarrow F_{d_\bullet}, \\ U_{d'_\bullet} &\mapsto (\mathbf{k}^{d_1-d'_1} \oplus U_{d'_1} \subset \dots \subset \mathbf{k}^{d_t-d'_t} \oplus U_{d'_t} \subset \mathbf{k}^{d_{t+1}-d'_{t+1}} \oplus \mathbf{k}^{d'_{t+1}} = W). \end{aligned}$$

There are t tautological bundles \mathcal{U}_{d_j} and t universal quotient bundles \mathcal{N}_j on F_{d_\bullet} , with $\mathrm{rk}(\mathcal{U}_{d_j}) = d_j$, $\mathrm{rk}(\mathcal{N}_j) = \nu_j$, for $j = 1, \dots, t$.

As usual, throughout the section, \mathcal{V} is a vector bundle on F_{d_\bullet} and $\mathcal{E} := \mathrm{End}(\mathcal{V})$.

Theorem 6.4 Assume that the sequence d_\bullet satisfies $d_\bullet \geq 2_\bullet$. Then the vector bundle \mathcal{V} on F_{d_\bullet} splits if and only if it does along some $\iota(F_{2_\bullet})$.

Everything is defined over a countable, algebraically closed field, which is simultaneously a subfield of \mathbf{k} and of \mathbb{C} (cf. Section 4). Using twice the invariance of the splitting under base change (cf. Proposition 1.7), we may—and we henceforth do—assume that $\mathbf{k} = \mathbb{C}$. The proof of the statement requires a few intermediate results.

Lemma 6.5 *With the previous notation, assume that $d_{j+1} - d_{j-1} \geq 3$, for all $j = 1, \dots, t$ (e.g. $d_\bullet \geq 2_\bullet$). Then F_{d_\bullet} is 1-splitting (cf. Definition 5.3).*

Proof. Any line bundle on F_{d_\bullet} can be written in the form

$$\ell = \det(\mathcal{U}_{d_1})^{-a_1} \otimes \dots \otimes \det(\mathcal{O}_F^{d_{t+1}}/\mathcal{U}_{d_t})^{-a_t}, \text{ with } a_1, \dots, a_t \in \mathbb{Z}.$$

It corresponds to the weight $\alpha = (\underbrace{a_1, \dots, a_1}_{d_1 \text{ times}}, \underbrace{a_2, \dots, a_2}_{d_2 - d_1 \text{ times}}, \dots, \underbrace{a_t, \dots, a_t}_{d_{t+1} - d_t \text{ times}}) \in \mathbb{Z}^{d_{t+1}}$.

Let $\rho := (d_{t+1} - 1, \dots, 1, 0)$. By Bott's theorem, $H^1(F, \ell) \neq 0$ if and only if

$$\alpha + \rho = (\underbrace{a_1 + d_{t+1} - 1, \dots, a_1 + d_{t+1} - d_1}_{\text{length}=d_1}, \dots, \underbrace{a_t + d_{t+1} - d_t - 1, \dots, a_t}_{\text{length}=d_{t+1} - d_t})$$

is *non-singular* (its entries are pairwise distinct), and it contains *exactly one inversion* (for the decreasing order). The t blocks which compose $\alpha + \rho$ are strictly decreasing and the only way to simultaneously achieve the previous two conditions is to have two consecutive blocks of length one each, that is $d_{j_0+1} - d_{j_0} = d_{j_0} - d_{j_0-1} = 1$ for some j_0 . \square

For $1 \leq j \leq t+1$, let $d_\bullet^j := (d_1, \dots, d_{j-1}, d_j - 1, \dots, d_t - 1; d_{t+1} - 1)$. We denote

$$S := \{(e, \eta) \in W \oplus W^\vee \mid \eta(e) \neq 0\}.$$

An element $s = (e, \eta) \in S$ determines a section in $\mathcal{N}_j \oplus \mathcal{U}_{d_{j-1}}^\vee$, which is globally generated, whose zero locus is

$$Y_s = Y_{(e, \eta)} = \{U_\bullet \in F_{d_\bullet} \mid U_{d_{j-1}} \subset \text{Ker}(\eta), e \in U_{d_j}\} \cong F_{d_\bullet^j}. \quad (6.7)$$

Therefore we have the same situation as in the section 4:

$$\begin{array}{ccc} & \mathcal{Y} & \\ \pi \swarrow & & \searrow q \\ S & & F_{d_\bullet}, \end{array} \quad (6.8)$$

where \mathcal{Y} is the zero locus of the universal section in $\mathcal{N}_j \oplus \mathcal{U}_{d_{j-1}}^\vee$ on $S \times F_{d_\bullet}$.

Lemma 6.6 *Assume that $d_\bullet \geq 2_\bullet$ and there is an index $1 \leq j \leq t+1$ such that $d_j - d_{j-1} \geq 3$. Then the cohomological dimension of $F_{d_\bullet} \setminus F_{d_\bullet^j}$ satisfies: $\text{cd}(F_{d_\bullet} \setminus F_{d_\bullet^j}) \leq \dim F_{d_\bullet} - 3$.*

Proof. For $Y_{(e, \eta)}$ as in (6.7), we have:

$$\begin{aligned} O := F_{d_\bullet} \setminus Y_{(e, \eta)} &= (F_{d_\bullet} \setminus \underbrace{\{U_\bullet \mid e \in U_{d_j}\}}_{=: F'}) \cup (F_{d_\bullet} \setminus \underbrace{\{U_\bullet \mid U_{d_{j-1}} \subset \text{Ker}(\eta)\}}_{=: F''}) \\ &= O' \cup O''. \end{aligned}$$

Note that $F' = \text{pr}_{\text{Gr}}^{-1}(\text{Gr}(d_j - 1; d_{t+1} - 1))$, where $\text{pr}_{\text{Gr}} : F_{d_\bullet} \rightarrow \text{Gr}(d_j; d_{t+1})$ is the natural projection. The Leray spectral sequence for pr_{Gr} and (6.1) imply that $\text{cd}(O') = \text{cd}(F_{d_\bullet} \setminus F') \leq \dim F_{d_\bullet} - d_j$. This proves the lemma for $j = 1$.

By using the duality $\text{Fl}(d_1, \dots, d_{t+1}) \cong \text{Fl}(d_{t+1} - d_t, \dots, d_{t+1})$ we deduce $\text{cd}(O'') \leq \dim F_{d_\bullet} - (d_{t+1} - d_{j-1})$. This proves the lemma for $j = t+1$.

Now assume $2 \leq j \leq t$. The morphism $\text{pr} : F_{d_\bullet} \rightarrow G := \text{Fl}(d_{j-1}, d_j; d_{t+1})$ is smooth, projective, and $Y_{(e,\eta)}$ is the pre-image of the analogous $Y'_{(e,\eta)} \subset G$. The Leray spectral sequence for $F_{d_\bullet} \setminus Y_{(e,\eta)} \rightarrow G \setminus Y'_{(e,\eta)}$ shows that it is enough to prove that $\text{cd}(G \setminus Y'_{(e,\eta)}) \leq \dim G - 3$. Henceforth, we assume that

$$d_\bullet = (0 < a < b < d), \text{ with } a \geq 2, b - a \geq 3, d - b \geq 2.$$

For a coherent sheaf \mathcal{G} on O , we have the Mayer-Vietoris sequence

$$\dots \rightarrow H^{i-1}(O' \cap O'', \mathcal{G}) \rightarrow H^i(O, \mathcal{G}) \rightarrow H^i(O', \mathcal{G}) \oplus H^i(O'', \mathcal{G}) \rightarrow \dots$$

The previous discussion shows that $\text{cd}(O'), \text{cd}(O'') \leq \dim F_{d_\bullet} - 3$, so it is enough to prove $\text{cd}(O' \cap O'') \leq \dim F_{d_\bullet} - 4$.

The pair (e, η) decomposes $W = \text{Ker}(\eta) \oplus \langle e \rangle$; let $\pi : W \rightarrow \text{Ker}(\eta)$ be the projection. Since $O' \cap O'' = \{U_\bullet \mid U_a \not\subset \text{Ker}(\eta), e \notin U_b\}$, we deduce the morphism:

$$f : O' \cap O'' \rightarrow G' := \text{Fl}(a-1, b; \text{Ker}(\eta)), \quad [U_a \subset U_b] \mapsto [U_a \cap \text{Ker}(\eta) \subset \pi(U_b)].$$

Claim 1 The fibre over $[V_{a-1} \subset V_b] \in G'$ is isomorphic to $(\frac{V_b}{V_{a-1}})^\vee \setminus \{0\}$; its cohomological dimension equals $b - a$.

Indeed, let $[U_a \subset U_b]$ be in the fibre. Since $\pi : U_b \rightarrow V_b$ is an isomorphism, U_b is the graph of a (uniquely defined) homomorphism $h : V_b \rightarrow \langle e \rangle$:

$$U_b = \{(v, h(v)) \mid v \in V_b\}.$$

Also, we have $U_a = V_{a-1} + \mathbb{C} \cdot (v_0, e)$, with $v_0 \in \pi(U_a) \subset \pi(U_b) = V_b$. The inclusion $U_a \subset U_b$ implies $h(V_{a-1}) = 0$, $h(v_0) = e$, hence $h \in (\frac{V_b}{V_{a-1}})^\vee \setminus \{0\}$. Conversely, any such h defines a flag $[U_a \subset U_b] \in O' \cap O''$.

Claim 2 $\text{cd}(O' \cap O'') \leq \dim F_{d_\bullet} - 4$.

For a quasi-coherent sheaf \mathcal{G} on $O' \cap O''$, the previous step implies that $R^{>(b-a)} f_* \mathcal{G} = 0$. Leray's spectral sequence yields: $\text{cd}(O' \cap O'') \leq (b - a) + \dim G' = \dim F_{d_\bullet} - (b - a + 1)$. \square

Lemma 6.7 *Assume that $d_\bullet \geq 2_\bullet$ and $d_j - d_{j-1} \geq 3$ for some $1 \leq j \leq t+1$. We assume that there is $o \in S$ such that \mathcal{Y}_{Y_o} splits. Then there is an open ball $B \subset S$, such that $(q^* \mathcal{V})_B$ splits.*

Proof. The claim follows from the Kodaira-Spencer deformation theory. Since $d_\bullet^j \geq 2_\bullet$, the Lemma 6.5 implies that Y_o is 1-splitting. The fibration $\mathcal{Y} \rightarrow S$ is locally trivial (in the analytic topology) with fibres isomorphic to $F_{d_\bullet^j}$, hence $q^* \mathcal{V}$ is locally an analytic family of vector bundles on $Y_o \cong F_{d_\bullet^j}$.

Since \mathcal{E}_{Y_o} splits, $H^1(Y_o, \mathcal{E}_{Y_o}) = 0$, therefore \mathcal{Y}_{Y_o} is rigid. Hence there is a ball $B \subset S$, such that $\mathcal{Y}_s \cong \mathcal{Y}_{Y_o}$, for $s \in B$ (cf. [11, Theorem 7.4], [9, Theorem 2.7]). Possibly after shrinking B , the splitting of \mathcal{Y}_{Y_o} extends over \mathcal{Y}_B . \square

Proof. (of the Theorem 6.4) We prove by induction on $t + \sum_{j=1}^{t+1} d_j$. For $t = 1$ and $\text{rk}(\mathcal{V})$ arbitrary, it is the Theorem 6.1. For the inductive step, let j be minimal such that $d_j - d_{j-1} \geq 3$. Since $F_{2_\bullet} \subset F_{d_\bullet^j}$, the induction hypothesis implies that \mathcal{V} splits along $Y_o = \iota(F_{d_\bullet^j}) \subset F_{d_\bullet}$. By the Lemma 6.7, there is an open ball $o \in B \subset S$ such that $(q^* \mathcal{V})_B$ splits.

Claim The gluing procedure in the Lemma 4.4 applies, so there is an open analytic neighbourhood \mathcal{U} of Y_o such that $\mathcal{Y}_{\mathcal{U}}$ is a successive extension of line bundles on \mathcal{U} . Indeed, as

we pointed out in the Remark 4.5, the proof of the Lemma 4.4 is based on the following assumptions:

- (i) the diagram (4.4)—which compares the Picard groups of $F_{d_\bullet}, Y_s, Y_{st} = Y_s \cap Y_t$, for general $s, t \in S$ —should consist of isomorphisms;
- (ii) the triple intersections $Y_{ost} = Y_o \cap Y_s \cap Y_t$ should be connected.

We verify that these conditions are fulfilled. For (i), take $s = (e, \eta), s' = (e', \eta')$ generic, so the intersection Y_{st} is transverse, and observe that:

$$\begin{aligned} Y_{st} &= \{ U_\bullet \mid e, e' \in U_{d_j} \text{ and } U_{d_{j-1}} \subset \text{Ker}(\eta) \cap \text{Ker}(\eta') \} \\ &\cong \text{Fl}(d_1, \dots, d_{j-1}, d_j - 2, \dots, d_{t+1} - 2). \end{aligned}$$

Clearly, the restrictions $\text{Pic}(F_{d_\bullet}) \rightarrow \text{Pic}(Y_s) \rightarrow \text{Pic}(Y_{st})$ are isomorphisms. For (ii), note that $Y_{ost} \cong \text{Fl}(d_1, \dots, d_{j-1}, d_j - 3, \dots, d_{t+1} - 3)$ is connected, even for $d_{j-1} = d_j - 3$.

Let \hat{F} be the formal completion of F_{d_\bullet} along Y_o . The claim implies that $\mathcal{V}_{\hat{F}}$ is a successive extension of line bundles. But $\text{cd}(F_{d_\bullet} \setminus Y_o) \leq \dim F_{d_\bullet} - 3$, by the Lemma 6.6, so the following are isomorphisms, cf. [8, Theorem III.3.4(b)]:

$$0 = \text{Ext}^1(\ell, \ell') \xrightarrow{\cong} \text{Ext}^1(\ell_{\hat{F}}, \ell'_{\hat{F}}), \forall \ell, \ell' \in \text{Pic}(F_{d_\bullet}), \Gamma(F_{d_\bullet}, \mathcal{E}) \xrightarrow{\cong} \Gamma(\hat{F}, \mathcal{E}_{\hat{F}}).$$

We deduce that $\mathcal{V}_{\hat{F}}$ splits and consequently \mathcal{V} splits too, by the Lemma 1.6. \square

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